NCU Ph.D Qualify Exam, Real Analysis, 中央大學博士資格考–實變

February 2025

1. **(15ps)** For any $x \in \mathbb{R} \setminus \mathbb{Q}$ and $\epsilon > 0$, there is $r \in \mathbb{Q}$ with $|x - r| < \epsilon$.

2. **(10ps)** Prove that for any $b > 0$ there is a non-measurable set $\mathcal{N} \subset \mathbb{R}$ with $|\mathcal{N}|_e = b$ where $|\cdot|_e$ denotes the Lebesgue outer measure of a set.

3. **(15ps)** Prove that if $f \in L^1([0, 2\pi])$, we have $\lim_{\lambda \to \infty} \int_0^{2\pi} f(x)\, sin(\lambda x)\, dx = 0$.

4. **(15ps)** Let $E_k \subset [0, 1]$ be a sequence of measurable sets such that for any finite set of indices $i_1, i_2, ..., i_k$

$$|E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}| = |E_{i_1}|\, |E_{i_2}|\, \cdots\, |E_{i_k}| .$$

   Prove that if $\sum_{k=1}^{\infty} |E_k| = \infty$ then $|limsup\ E_k| = 1$, where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}$.

5. **(15ps)** Compute $\lim_{n \to \infty} n^3 \int_n^{2n} \frac{x}{x^5 + 1}\, dx$.

6. **(15ps)** Either prove or give a counterexample to the following statement. If E is measurable and $|E| < \infty$ then $\forall\, \epsilon > 0 \ : \ \exists\, K \subseteq E$ where K is **compact** and $|E \setminus K| < \epsilon$.

7. **(15ps)** Suppose $f \in L^p([0, \infty))$, $f \geq 0$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Let

$$F(x) = \frac{1}{x} \int_0^x f(t)\, dt, \ x > 0 .$$

   Prove that $\|F\|_p \leq q\, \|f\|_p$.