中央大學博士資格考--實變函數論–NCU Ph.D Qualify Exam, Real Analysis.

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Keep in mind that both the correctness and quality of proofs are important. All sets and functions are assumed to be Lebesgue measurable unless said otherwise. The notations $|\cdot|$, $|\cdot|_e$ denote the Lebesgue and Lebesgue outer measure respectively.

1. **(15ps)** Suppose $g_k, g \in L^p(E)$ where $1 < p \leq \infty$ satisfying $g_k \longrightarrow g$ $a.e.$ and there exists $M > 0 : \forall k : \|g_k\|_p \leq M < \infty$. Prove that for all $f \in L^q(E)$ where $\frac{1}{p} + \frac{1}{q} = 1$ we have $\lim_{k\to\infty} \int_E g_k f \;=\; \int_E g f$.

2. **(10ps)** Let $E \subseteq \mathbb{R}^n$ be **any** set. Prove that E is the set of all continuous points of a function $f : \mathbb{R}^n \to \mathbb{R}$ if and only if E is a G_δ set.

3. **(15ps)** For $0 < p < 1$ and q satisfying $\frac{1}{p} + \frac{1}{q} = 1$ or $0 < q < 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $f, g : E \to \overline{\mathbb{R}}$, prove that
$$\int_E f\, g \;\geq\; \left(\int_E f^p\right)^{\frac{1}{p}} \cdot \left(\int_E g^q\right)^{\frac{1}{q}} .$$

4. **(15ps)** Either prove or disprove that **any** set $A \subseteq \mathbb{R}^n$ is contained in some Lebesgue measurable set E with $|A|_e = |E|$.

5. **(15ps)** Let $f_k : E \to \overline{\mathbb{R}}$ and $\exists\, \phi \in L(E) : \forall\, k : |f_k| \leq \phi$ $a.e.$. Suppose $f_k \xrightarrow{m} f$, prove that $f \in L(E)$ and
$$\lim_{k\to\infty} \int_E f_k \;=\; \int_E f\ .$$

6. **(15ps)** Suppose $f \in L^\infty([a,b])$ and $\lim_{x\to a^+} f(x) = L$ satisfying $0 < L < \infty$. Does the following limit exist? If yes then compute the limit, if not give a reason why it does not exist.
$$\lim_{t\to a^+} \int_a^t \frac{f(x)}{\sqrt{(x-a)\cdot(t-x)}}\, dx\ , \quad t \in [a,b]\ .$$

7. **(15ps)** Suppose $E \subseteq \mathbb{R}^n$ (not necessarily measurable) satisfies $|E|_e > 0$. Prove that $\forall\, \delta \in (0, |E|_e)\ :\ \exists\, K \subseteq E$ where K is a compact set and $|K| = \delta$.