Ph.D Qualify Exam. Real Analysis

Feb 2021

1. **(15ps)** Give an example of a function $f : [\pi, \infty) \to \mathbb{R}$ that is improperly Riemann integrable but $f \notin L^1([\pi, \infty), dx)$, that is **not** Lebesgue integrable.

2. **(15ps)** Show that there is a measurable set $E \subset \mathbb{R}$ with $|E| > 0$, E contains no intervals.

3. **(15ps)** Suppose $f_k, f : E \to \overline{\mathbb{R}}$ are measurable functions, $p > 0$ and $\lim_{k\to\infty} \int_E |f - f_k|^p \, dx \longrightarrow 0$. Show that there is a subsequence $\{f_{k_j}\}$ for which $f_{k_j} \longrightarrow f$ *a.e.* in E.

4. **(20ps)** Let $f : [a, b] \to \mathbb{R}$ be bounded. Prove that f is Riemann integrable if and only if f is continuous *a.e.* on $[a, b]$.

5. **(20ps)** Let $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$ be a sequence of non-empty sets in $\mathbb{R}^n$ and let $E = \bigcup_{j=1}^{\infty} E_j$. Prove that $\lim_{j\to\infty} |E_j|_e = |E|_e$.

6. **(15ps)** Given $E \subseteq \mathbb{R}^n$ a measurable set with $|E| < \infty$. Show that for every $\epsilon > 0$, there exists a finite union of (closed) intervals $F = \bigcup_{j=1}^{N} Q_j$ such that

$$|E \,\Delta\, F| \leq \epsilon\,, \quad \text{where } E \,\Delta\, F \text{ denotes the symmetric difference } (E \setminus F) \cup (F \setminus E).$$