NCU Ph.D Qualifying Exam 中央大學博士資格考. Real Analysis 實變函數論

Fall 2021

1. **(20%)** Let $f, f_n \in L^p(E), g, g_n \in L^q(E)$ where E is measurable and $\frac{1}{p}+\frac{1}{q}=1$, $p,q \geq 1$. Suppose $\|f_n - f\|_{L^p(E)}$, $\|g_n - g\|_{L^q(E)} \longrightarrow 0$ as $n \to \infty$. Prove that

$$\lim_{n\to\infty} \int_E |f_n g_n - f g| = 0 .$$

2. **(20%)** E is a measurable set and $f \in L^r(E)$ for some $r < \infty$. Prove that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

3. **(20%)** Let $f : [0,1] \to [0,\infty)$ by

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ min\{k \mid a_k \neq 0\} - 1 & x = 0.a_1a_2a_3a_4... \notin \mathbb{Q} , \end{cases}$$

where $0.a_1a_2a_3a_4...$ is the decimal expansion of $x \in [0,1]$. For example $f(0.00321450078...) = 2$, $f(0.0700340098...) = 1$, $f(0.00000334789...) = 5$ etc... Show that f is measurable and find the Lebesgue integral

$$\int_{[0,1]} f .$$

4. **(10%)** Either prove or disprove the following statement: If $\{a_n\}$ is a sequence of real numbers with $\lim_{n\to\infty} a_n = a$ then

$$\lim_{n\to\infty} \frac{a_{n+1}^{n+1}}{a_n^n} = a .$$

5. **(10%)** For $x \in [0,1]$ denotes by $0.a_1a_2a_3...$ the decimal representation of x. In the case x has two different representations such as $x = 0.34199999..... = 0.342$ we choose $x = 0.34199999....$ Define

$$B = \{x \in [0,1] : x = 0.a_1a_2a_3..., \exists n \in \mathbb{N} : a_n = 7\} .$$

Show that B is measurable and compute the Lebesgue measure of B.

6. **(20%)** Let $\mathbb{Q} = \{a_1, a_2, a_3, ...\}$. Define $G = \bigcup_{n=1}^{\infty} \left(a_n - \frac{1}{n^2} , a_n + \frac{1}{n^2} \right)$. For any closed set F prove that $m(F \Delta G) > 0$ where $A \Delta B = (A \setminus B) \cup (B \setminus A)$ and $m(\cdot)$ denotes the Lebesgue measure.