National Central University Ph.D Qualify Exam. Real Analysis
Aug 2020

1. **(5ps)** State the definition of the Lebesgue outer measure of a set $E \subseteq \mathbb{R}^n$.

2. **(20ps)** Let $|\cdot|_e$ denotes the Lebesgue outer measure. For any rectangle $R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ in $\mathbb{R}^n$ let $v(R) = (b_1 - a_1) \cdot (b_2 - a_2) \cdots (b_n - a_n)$ denotes its volume. Either prove that $|R|_e = v(R)$ or give a counterexample to show that this statement is false.

3. **(15ps)** Let $A, B \subseteq \mathbb{R}$ be any sets such that $A \cap B = \emptyset$. Either prove that $|A \cup B|_e = |A|_e + |B|_e$ or give a counterexample to show that this statement is false.

4. **(20ps)** If $|E| < \infty$ where $|\cdot|$ denotes the Lebesgue measure Prove

$$\lim_{p \to \infty} \left(\int_E |f(x)|^p \, dx \right)^{\frac{1}{p}} = \|f\|_{\infty, E} \, .$$

   In the above the integral is the Lebesgue integral and the norm $\|f\|_{\infty,E} = \underset{E}{ess\ sup} |f|$ and is defined to be

$$\underset{E}{ess\ sup}\, f = \begin{cases} \infty & \text{if } \forall\, \alpha \in \mathbb{R} \ : \ \omega(\alpha) > 0 \\ inf\{\alpha \in \mathbb{R} \,|\, \omega(\alpha) = 0\} & \text{otherwise.} \end{cases}$$

   where $\omega(\alpha) = |\{x \in E \,|\, f(x) > \alpha\}|$.

5. **(5ps)** Complete the following definition: Let Let $f, f_k : E \to \overline{\mathbb{R}}$ be finite *a.e.* and measurable functions, where $\overline{\mathbb{R}}$ denotes the extended reals. f_k is said to converge in measure (or in probability) on E to f denoted by $f_k \xrightarrow{m} f$ if .....

6. **(10ps)** Either prove or give a counterexample that if $f, f_k : E \to \overline{\mathbb{R}}$ be measurable and finite *a.e.* If $f_k \longrightarrow f$ *a.e.* on E then $f_k \xrightarrow{m} f$ on E.

7. **(10ps)** Show that a set $E \subset \mathbb{R}^n$ is measurable if and only if for every set $A \subset \mathbb{R}^n$ we have $|A|_e = |A \cap E|_e + |A - E|_e$.

8. **(15ps)** Let f be a continuous function on $[0, 1]$. Suppose for all $k = 0, 1, 2, ...$ $\int_0^1 x^k \, f(x) \, dx = 0$ then $\forall\, x \in [0, 1]$ $f(x) = 0$.