Department of Mathematics
National Central University

Ph. D. Qualifying Examination

Fall, 2000.

Real Analysis

1. Let f be a continuous function one-to-one from a compact space X onto a Hausdorff space Y. Prove that f is an homeomorphism.

2. Let X and Y be metric spaces. Let $\{f_n\}_n$ be a sequence of continuous functions from X into Y which converge to a function f uniformly on each compact subset of X. Show that f is continuous.

3. Let X and Y be compact Hausdorff spaces. Show that for each continuous real-valued function f on $X \times Y$ and each $\epsilon > 0$, there are continuous real-valued functions $g_1, \ldots, g_n$ on X and $h_1, \ldots, h_n$ on Y such that for each $(x, y) \in X \times Y$ we have

$$\left| f(x,y) - \sum_{i=1}^{n} g_i(x) h_i(y) \right| < \epsilon.$$

4. Prove that if a real-valued function f is integrable on $[a, b]$ and

$$\int_a^x f(t)\, dt = 0$$

for all x in $[a, b]$ then $f(t) = 0$ a.e. in $[a, b]$.

5. (a) Show that the set $\mathbb{R}^{n-1} \times \{0\}$ has measure zero in $\mathbb{R}^n$.

   (b) Let $[0,1]^2 = [0,1] \times [0,1]$. Let $f : [0,1]^2 \longrightarrow \mathbb{R}$ be defined by setting $f(x, y) = 0$ if $y \neq x$ and $f(x, y) = 1$ if $y = x$. Show that f is integrable over $[0,1]^2$.

6. Prove that if a subset A of $[0,1]$ has measure zero then

$$A^2 = \{x^2 : x \in A\}$$

has measure zero, too.

7. Suppose that f is a real-valued differentiable function on $[0,1]$. Prove that its derivative function f' is Lebesque measurable on $[0,1]$.

8. Let μ_n be a non-decreasing sequence of measures defined on a measurable space $(X, \mathcal{A})$ in the sense that $\mu_n(A) \uparrow \mu(A)$ for all A in $\mathcal{A}$.

   (a) Prove that μ is a measure on $(X, \mathcal{A})$ with respect to which all μ_n are absolutely continuous.

   (b) On any fixed set A of finite μ measure, let f_n denote the Radon–Nikodym derivative of μ_n with respect to μ. Prove that almost everywhere (with respect to μ, and thus all μ_n) on A, $f_n \uparrow 1$.

9. Suppose that f is a real-valued continuous function on $[0,1]$. Show that $\|f\|_{L^p} \longrightarrow \|f\|_{L^\infty}$ as $p \longrightarrow \infty$.