Probability Qualify Examination

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1. Let $\mathcal{F}$ be a σ-algebra on Ω and let δ be a π-system with $\sigma(\delta)=\mathcal{F}$. If P and Q are probabilities on $\mathcal{F}$, such that P=Q on δ, show that P=Q on $\mathcal{F}$.

2. Let X and Y be random variables and let A be an event. Prove that the function $Z = XI_A + YI_{A^C}$ is a random variable.

3. Prove that for $X \geq 0$, $\sum_{k=1}^{\infty} P\{X \geq k\} \leq E(X) \leq \sum_{k=0}^{\infty} P(X \geq k)$.

4. Let X be a P(λ) (Poisson distribution with parameter λ), show that $P(X \geq 2\lambda) \leq (e/4)^{\lambda}$.

5. Let $X_1, X_2, \ldots$ be pairwise uncorrelated with mean 0 and partial sums $S_n = \sum_{k=1}^{n} X_k$. Prove that if there is a constant c such that $Var(X_k) \leq c$ for every k, then $\frac{S_n}{n^{\alpha}} \xrightarrow{q.m.} 0$ for all $\alpha > 1/2$.

6. Let $X_1, X_2, \ldots$ be i.i.d. and integer-valued, with partial sums $S_n = \sum_{i=1}^{n} X_i$. Prove that for each n and k, $P(S_n = k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikt}\phi_X(t)^n dt$, where $\phi_X(t)$ is the characteristic function of X.

7. Let $X_1, X_2, \ldots$ be independent random variables, $\sigma_n^2 = \mathrm{var}(X_n)$.

   If there is a constant c such that $P\{|X_k| \leq c\} = 1$ for each k and if $\sigma_n^2 \to \infty$ as $n \to \infty$. Show that the Lindeberg condition is satisfied.

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8. Let $X_1, X_2, \ldots$ be i.i.d. with $E(X_1) = \mu, \text{var}(X_1) = \sigma^2$ and finite fourth central moment $\mu_4 = E[(X_1 - \mu)^4]$. Define the sample variance $S_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$, where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ is the sample mean. Prove that

$$\frac{\sqrt{n}(S_n^2 - \sigma^2)}{\sqrt{\mu_4 - \sigma^4}} \xrightarrow{d} N(0,1).$$

9. Let $X, Y \in L^2$ be such that $E(X \mid Y) = Y$ and $E(Y \mid X) = X$.

Prove that $X \overset{a.s.}{=} Y$.

10. Let X be a submartingale with respect to Y. Prove that for each n and each $a > 0$,

$$P\{\max_{1 \le k \le n} X_k \ge a\} \le \frac{1}{a} E(X_n; \{\max_{1 \le k \le n} X_k \ge a\}) \le \frac{E(X_n^+)}{a}.$$