歸檔

# 國立中央大學數學系
# 博士班資格考試
## 〈機率論〉試題
## 2002 年 2 月

考試時間三小時。共計十題，每題十分。

(1) Let $F_0$ be a field and $G$ be the minimal M.C. (Monotone Class) containing $F_0$. Define two classes of sets as follows:
$C_1 = \{E \in G : E \cap F \in G \text{ for all } F \in F_0\}$,
$C_2 = \{E \in G : E \cap F \in G \text{ for all } F \in G\}$.
Show that both $C_1$ and $C_2$ are M.C.'s and equal to $G$.

(2) Let $\{X_n\}$ be a sequence of identically distributed random variables with finite mean. Show that $\lim_n \frac{1}{n} E\{\max_{1\le j\le n}|X_j|\} = 0$.

(3) Show that $\{X_n\}$ converges in probability to $X$ if and only if for every subsequence $\{X_{n_k}\}$ contains a further subsequence that converges almost surely to $X$.

(4) Let $\{X_n\}$ be a sequence of random variables such that $\sup_n E(|X_n|^p) < \infty$ for some $p > 1$. Show that $\{X_n\}$ is uniformly integrable.

(5) Let $\{X_n\}$ be a sequence of independent, identically distributed random variables taking values $\pm 1$ with probability $\frac{1}{2}$ each. Find the range of $\theta$ such that $\sum_n \frac{X_n}{n^\theta}$ converges almost surely, and give the explanation.

(6) Let $\{X_n\}$ be a sequence of independent, identically distributed random variables with $EX_n = 0$ and $EX_n^2 = \sigma^2$. Show that $\dfrac{\sum_{i=1}^n X_i}{\left(\sum_{i=1}^n X_i^2\right)^{\frac{1}{2}}}$ converges in distribution to the standard normal distribution.

(7) Let $\{X_{nj}, 1 \le j \le k_n\}$ be a double array of rowwise independent random variables, where $k_n \to \infty$ as $n \to \infty$. Suppose that for each n and j there is a finite constant $M_{nj}$ such that $|X_{nj}| \le M_{nj}$ and that $\max_{1 \le j \le k_n} M_{nj} \to 0$. Show that $\frac{S_n - ES_n}{\sigma_n}$ converges in distribution to the standard normal distribution, where $S_n = X_{n1} + X_{n2} + ... + X_{nk_n}$ and $\sigma_n^2 = \{\text{Van}(S_n)\}$.

(8) Let $\{X_n\}$ be a sequence of independent, identically distributed random variables and $N(\lambda)$ be an independent Poisson random variables with mean $\lambda$. Show that $W = X_1 + X_2 + ... + X_{N(\lambda)}$ has an infinitely divisible distribution.

(9) Let X be a random variable on $\Omega$ the probability space, let $A_1, A_2, ..., A_n$ be a partition of $\Omega$ into disjoint sets each of which has positive probability, and let $F = \sigma(A_1, A_2, ..., A_n)$ be the $\sigma$ field of generated by these sets. Find $E(X|F)$ the conditional expectation of $X$ given $F$, and give the explanation.

(10) Let $\xi_i^n, i, n \ge 0$ be independent, identically distributed nonnegative integer valued random variables. Define a sequence $Z_n$, $n \ge 0$ by $Z_0 = 1$ and $Z_{n+1} = \xi_1^{n+1} + \xi_2^{n+1} + ... + \xi_{Z_n}^{n+1}$ if $Z_n > 0$, and $Z_{n+1} = 0$ if $Z_n = 0$.

Let $F_n = \sigma(\xi_i^m : i \ge 1, 1 \le m \le n)$ be the $\sigma$ field of generated by these random variables and $\mu = E\xi_i^m$. Show that $\frac{Z_n}{\mu^n}$ is a martingale with respect to $F_n$.