歸檔

國立中央大學數學系

博士班資格考試

＜機率論＞試題

2002年9月

1. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and $\mathcal{N}_0$ be the set of all null sets in $(\Omega, \mathcal{F}, \mathcal{P})$. Show that for any Borel subfield $\mathcal{F}_1$ of $\mathcal{F}$, there exists a minimal Borel field $\mathcal{F}_2$ satisfing $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}$ and $\mathcal{N}_0 \subset \mathcal{F}_2$.

   (A set E belongs to $\mathcal{F}_2$ if and only if there exists a set F in $\mathcal{F}$, such that $E \triangle F \in \mathcal{N}_0$.)

   (*10%*)

2. Let Y be a nonnegative random variable and $k > 0$. Show that $E(Y^k) = \int_0^\infty k y^{k-1} \mathcal{P}(Y > y) dy$. (*10%*)

3. (a) What is the (1st) Borel-Cantelli Lemma? Prove or disprove the converse of the Borel-Cantelli lemma.

   (b) Show that $X_n \to X$ in probability if and only if for every subsequence $X_{n(m)}$ there is a further subsequence $X_{n(m_k)}$ that converges almost surely to X.

   (*15%*)

4. (a) What is the Kolmogorov's Three Series Theorem?

   (b) Show that if $X_1, X_2, \cdots$ are independent with $EX_n = 0$ and $\sum_{n=1}^{\infty} \left(X_n^2 I_{(|X_n| \le 1)} + |X_n| I_{(|X_n| > 1)} \right) < \infty$ then $\sum_i X_n$ converges.

   (*10%*)

5. (a) Let $X_1, X_2, \cdots$ be independent and let $\alpha_n = (Var\, S_n)^{1/2}$, where $S_n = X_1 + \cdots + X_n$. If there is a $\delta > 0$ so that $\lim_{n \to \infty} \alpha_n^{-(2+\delta)} \sum_{m=1}^{n} E(|X_m - EX_m|^{2+\delta}) = 0$, show that $(S_n - ES_n) / \alpha_n$ converges in distribution to the unit normal.

(b) If $X_1, X_2, \cdots$ are independent with $|X_i| \leq M$ and $\sum_{n=1}^{\infty} Var(X_n) = \infty$, show that there are constants a_n, b_n so that $(S_n - b_n)/a_n$ converges to the normal distribution.

(15%)

6. Show that

(a) $\mu(\{a\}) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-ita}\varphi(t)dt$, where $\varphi(t)$ is the characteristic function of the propability measure $\mu(\cdot)$.

(b) $\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\varphi(t)|^2 dt = \sum_{X\in R} \mu(\{X\})^2$.

(Hint:you may use (a) and consider the random variable $X - Y$, where X, Y are iid.)

(15%)

7. Suppose that X is an integrable random variable on $(\Omega, \mathcal{F}, \mathcal{P})$ and that $\mathcal{G}$ is sub-σ-field in $\mathcal{F}$. Show that (15%)

(i) there is a random variable $E_{\mathcal{G}}(X)$, called the conditional expectation of X given $\mathcal{G}$, which has the following two properties:

(a) $E_{\mathcal{G}}(X)$ is integrable w.r.t. $\mathcal{G}$, and

(b) $\int_G E_{\mathcal{G}}(X)d\mathcal{P} = \int_G X d\mathcal{P}$ for all $G \in \mathcal{G}$.

(ii) If $\mathcal{G}_1$ and $\mathcal{G}_2$ are σ-fields such that $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \mathcal{F}$, then $E_{\mathcal{G}_1}[E_{\mathcal{G}_2}(X)] = E_{\mathcal{G}_1}(X)$.

(iii) If $EX^2 < \infty$, then $E_{\mathcal{G}}(X)$ is the projection of X from $L^2(\mathcal{F})$ onto to $L^2(\mathcal{G})$, where $L^2(\mathcal{F}) = \{Y \in \mathcal{F} : EY^2 < \infty\}$.

8. (a) What is the Martingale Convergence Theorem?

(b) Show that if $X_n \geq 0$, $n \geq 0$, is a supermartingale, then $X_n \to X$ a.s. and $EX \leq EX_0$.

(10%)