Probability Theory

1. (14 points) Let X be a positive random variable with $\mathbb{E}[X^2] < \infty$.

   (a) (6 points) Show that
   $$a^2\mathbb{P}(X \geq a) \leq \mathbb{E}[X^2]$$
   for $a \geq 0$.

   (b) (8 points) Show that
   $$\mathbb{P}(X > 0) \geq \frac{(\mathbb{E}[X])^2}{\mathbb{E}[X^2]}.$$

2. (18 points) Let $f : [0,1] \longrightarrow \mathbb{R}$ be a continuous function and the random variables $X_1, X_2, ..., X_n$ follow the Bernoulli distribution with parameter p. Set
   $$f_n(p) = \mathbb{E}\left[f\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)\right].$$

   (a) (6 points) Find the value of $f_n(p)$.

   (b) (12 points) Show that $f_n \longrightarrow f$ uniformly on $[0,1]$.

3. (10 points) Suppose that X is a continuous random variable with distribution function F. What kind of distribution does $F(X)$ have? Justify your answer.

4. (16 points) Given the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, show that there exists a compete space $(\Omega, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ such that $\mathcal{F} \subset \bar{\mathcal{F}}$ and $\mathbb{P} = \bar{\mathbb{P}}$ on $\mathcal{F}$.

5. (18 points) Suppose $X_1, X_2, ...$ are independent and
   $$\mathbb{P}(X_m = m) = \mathbb{P}(X_m = -m) = \frac{m^{-2}}{2}, \qquad \mathbb{P}(X_m = 1) = \mathbb{P}(X_m = -1) = \frac{1 - m^{-2}}{2}.$$

   (a) (6 points) Find the value of $\lim_{n\to\infty} \frac{1}{n} \operatorname{Var}\left(\sum_{i=1}^{n} X_i\right)$.

   (b) (12 points) Show that $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$ converges weakly to standard normal distribution.

6. (24 points) Let Ω be the unit disc $\{(x, y) : x^2 + y^2 \leq 1\}$ with the σ-algebra of Borel sets and $\mathbb{P}$ the Lebesgue measure on the disc normalized so that $\mathbb{P}(\Omega) = 1$. Suppose that X and Y are the projections onto the x and y axes, respectively, i.e.,

$$X(x, y) = x, \qquad Y(x, y) = y$$

for any $(x, y) \in \Omega$.

(a) (4 points) For Borel set $A \subset \Omega$, find $\mathbb{P}(A)$.

(b) (8 points) What is the joint density function of X and Y?

(c) (12 points) Find $\mathbb{E}[X^2|Y]$.