# NCU Qualify Exam for Ordinary Differential Equations

09/01/2009

1. (a) (10 pts) Let $f(t,x)$ be continuous on

$$S = \{(t,x) : |t-t_0| \leq a,\ |x-x_0| \leq b\}$$

and let $f(t,x)$ satisfy a Lipschitz condition in $x$ with Lipschitz constant $L$. Let $M \equiv \max\{|f(t,x)| : (t,x) \in S\}$. Show by contraction principle that there exists a unique solution of initial value problem

$$\frac{dx}{dt} = f(t,x),$$
$$x(t_0) = x_0$$

on $I = \{t : |t-t_0| \leq \alpha\}$, where $\alpha < \min\{a, \frac{b}{M}, \frac{1}{L}\}$.

(b) (10 pts) Consider the equation

$$\dot{x} = \phi(t)\frac{x}{1+t^2+x^2} = f(t,x)$$

for all $t, x,\ \phi(t) > 0$, and $\int^{\infty} \phi(t)dt < \infty$. Show that every solution approaches a constant as $t \to \infty$.

2. (a) (10 pts) We consider Hill's equation

$$x'' + (\alpha + p(t))x = 0,$$

where $\alpha > 0$ and $p(t)$ is periodic with period $T$. Rewrite Hill's equation as a linear system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A(t)\begin{pmatrix} x \\ y \end{pmatrix}.$$

Let $\Phi(t)$ be the fundamental matrix of the linear system above. Also let

$$\Phi(0) = I,\ -2 < \mathbf{tr}(\Phi(t)) < 2.$$

Show that the solutions of Hill's equation are bounded.

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(b) (10 pts) Find the general solution of $x' = Ax$, where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

3. (a) (10 pts) Consider the following initial value problem

$$\begin{aligned} x' &= 3x\Big(1 - \frac{x}{2}\Big) - \frac{2x}{1+x}y \\ y' &= \Big(\frac{2x}{1+x} - 1\Big)y \\ x(0) &= 1.2,\ y(0) = 1.2 \end{aligned}$$

Please find $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$.

(b) (10 pts) Please use the following system

$$\begin{aligned} x' &= -x \\ y' &= y + x^2 \end{aligned}$$

to verify the Stable Manifold Theorem.
*Hint.* The system can be solved directly.

4. a. (10 pts) Consider the following system

$$\begin{aligned} x' &= y - x\Big(\frac{x^2 - xy + y^2}{x^4 + y^4}\Big) \\ y' &= -x - y\Big(\frac{x^2 - xy + y^2}{x^4 + y^4}\Big) \end{aligned}$$

Discuss the stability of equilibria.

b. (5 pts) Please show the Van der Pol equation

$$x'' = \varepsilon(x^2 - 1)x' + x = 0,\ \varepsilon > 0$$

is equivalent to the Lienard system

$$\begin{aligned} x' + y - \varepsilon\Big(\frac{x^3}{3} - x\Big) \\ y' = -x \end{aligned}$$

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(c) (10 pts) For the Lienard system in (b), please show by LaSalle's invariant principle that

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0$$

if $x^2(0) + y^2(0) < 3$.

5. (a) (5 pts) State the Poincaré-Bendixson Theorem.

(b) (10 pts) Show that, for all $\alpha > 0$, the system

$$x' = y$$
$$y' = -x + (\alpha - x^2 - y^2)y$$

has a unique stable limit cycle which is the $\omega$-limit set of every trajectory except the critical point at the origin.