Qualifying Exam.: Ordinary Differential Equations Feburary, 2020

1.(10%) Let $x(t)$, $f(t)$ and $g(t)$ be real continuous functions defined in $[a,b]$ such that $x(t) \geq 0$ for all $t \in [a,b]$. Suppose $f(t)$ is differentiable and

$$x(t) \leq f(t) + \int_a^t g(s)x(s)ds, \text{ for all } t \in [a,b].$$

Prove that

$$x(t) \leq f(a)\int_a^t g(s)ds + \int_a^t f'(s)\exp\Big\{\int_s^t g(u)du\Big\}ds.$$

2. (20%) (1) (10%) Consider the third order differential equation:

$$(2-t)y''' + (2t-3)y'' - ty' + y = 0, \quad t < 2.$$

Please find the general solution of the ODE by using the particular solution e^t.

(2) (10%) Find the general solution of the following non-homogeneous system:

$$\begin{cases} x' = 7x - y + 6z - 5t - 6, \\ y' = -10x + 4y - 12z - 4t + 23, \\ z' = -2x + y - z + 2. \end{cases}$$

3. (30%) (1) (10%) Please state the Poincare-Bendixson Theorem of ODE.

(2) (10%) Consider the system:

$$\begin{cases} x' = 3x + y - x(x^2+y^2), \\ y' = -x + 3y - y(x^2+y^2). \end{cases}$$

Does the system have periodic solutions or not ? If yes, please find them and determine their stability.

(3) (10%) Consider the system:

$$\begin{cases} x' = 2x + y + x^3, \\ y' = 3x - y + y^3. \end{cases}$$

Does the system have periodic solutions or not ? If yes, please find them and determine their stability.

4. (40%) (1) (10%) Please state the Floquet's Theorem of ODE.

(2) (10%) Suppose $\Phi(t) = \begin{bmatrix} e^{-2t}\cos t & -\sin t \\ e^{-2t}\sin t & \cos t \end{bmatrix}$ is a fundamental matrix of the linear system $\dot{\mathbf{x}} = A(t)\mathbf{x}$. Please find $A(t)$ and solution of the initial value problem:

$$\dot{\mathbf{x}} = A(t)\mathbf{x} + (1, e^{-2t})^T, \quad \mathbf{x}(0) = \mathbf{x}_0.$$

(3) (10%) Consider the ODE system:

$$\begin{cases} x' = x - y - x^3 - xy^2, \\ y' = x + y - x^2y - y^3, \\ z' = \lambda z, \end{cases}$$

where λ is a real constant. Please find the Poincaré map for the periodic solution $\gamma(t) := (\cos t, \sin t, 0)$, and the Floquet's exponent of $\gamma(t)$.

(4) (10%) Let $a(t)$ and $b(t)$ be continuous and T-periodic functions, $\phi_1(t)$ and ϕ_2(t) be solutions of the ODE: $y'' + a(t)y' + b(t)y = 0$. Assume that $\phi_1(0) = 1$, $\phi_2(0) = 0$, $\phi_1'(0) = 0$, $\phi_2'(0) = 1$. Please find the Floquet's multipliers of the ODE.