Ordinary Differential Equations

2019-01

1. Consider the initial value problem (IVP)

$$\begin{cases} \frac{dx}{dt} = f(t,x), f : D \to \mathbb{R}^n, \\ x(t_0) = x_0, \end{cases}$$

where D is an open set in $\mathbb{R} \times \mathbb{R}^n$, (t_0, x_0) is in D.
Prove that (A) if f is continuous in D and satisfies Lipschitz condition in x , then above (IVP) has one and only one solution on the interval $I = [t_0 - c, t_0 + c]$ for some $c > 0$.
(B) If, in addition, f is differentiable in x then the unique solution $x(t, x_0)$ is differentiable with respect to x_0.

2. Do the stability analysis for the following Lotka-Volterra two species competition model Does the model have a periodic solution in some parameters ? State and show your answer.

$$\begin{cases} \frac{dx_1}{dt} = c_1 x_1 (1 - \frac{x_1}{d_1}) - \beta_1 x_1 x_2, \\ \frac{dx_2}{dt} = c_2 x_2 (1 - \frac{x_2}{d_2}) - \beta_2 x_1 x_2, \\ x_1(0) > 0, x_2(0) > 0, \end{cases}$$

where $c_1, c_2, d_1, d_2, \beta_1, \beta_2 > 0$.

3. Consider the Lotka-Volterra two species competition model

$$\begin{cases} \frac{dx_1}{dt} = r_1 x_1 (1 - \frac{x_1}{K_1}) - \alpha x_1 x_2, \\ \frac{dx_2}{dt} = r_2 x_2 (1 - \frac{x_2}{K_2}) - \beta x_1 x_2, \\ x_1(0) > 0, x_2(0) > 0, \end{cases}$$

Show the solution $x_1(t) > 0, x_2(t) > 0$ for all $t > 0$, and x_1 and x_2 are all bounded. Does the model have a periodic solution in some

parameters ? State and show your answer.

4. Consider the two-dimensional system $X' = (A+B(t))X$, where

$$A = \begin{pmatrix} -1 & 0.5 \\ 0.5 & -1 \end{pmatrix},$$

$$B(t) = \begin{pmatrix} e^{-t^2} & \frac{1}{1+t^2} \\ \frac{1}{1+t^{2002}} & e^{-t} \end{pmatrix},$$

Prove that $\lim_{t\to\infty} X(t) = 0$ for every solution $X(t)$.

5. Show that $x'' + f(x)x' + g(x) = 0$ cannot have any periodic solution in a region where $f(x)$ is of the same sign.