Qualifying Exam for Numerical Analysis
Department of Mathematics, National Central University
Spring 2023

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1. (10%) Let $A \in \mathbb{R}^{m\times n}$ with $m > n$, $\mathbf{x} \in \mathbb{R}^n$, and $\mathbf{b} \in \mathbb{R}^m$. Define the regularized least squares error by

$$E(\mathbf{y}) = \|\mathbf{b} - A\mathbf{y}\|_2^2 + \lambda\|\mathbf{y}\|_2^2$$

for $\mathbf{y} \in \mathbb{R}^n$ and $\lambda > 0$. Show that if $\mathbf{x}$ minimizes the regularized least squares error, then $\mathbf{x}$ solves the equation $(A^TA + \lambda I)\mathbf{x} = A^T\mathbf{b}$.

2. (20%) Consider the rectangular linear system $A\mathbf{x} = \mathbf{b}$, where $A \in \mathbb{R}^{m\times n}$ with $m > n$ and $\mathbf{b} \in \mathbb{R}^m$. Suppose A is full rank with singular values $\sigma_1 \geq \sigma_2 \geq \cdots > \sigma_n > 0$, and the singular value decomposition of A is given by $A = U\Sigma V^T$, where $U \in \mathbb{R}^{m\times m}$, $\Sigma \in \mathbb{R}^{m\times n}$, and $V \in \mathbb{R}^{n\times n}$.

   (a) Derive the condition number $\text{cond}_2(A^TA)$.

   (b) The pseudo inverse of Σ is defined by $\Sigma^\dagger = (\Sigma^T\Sigma)^{-1}\Sigma^T$; similarly, the pseudo inverse of A is defined by $A^\dagger = (A^TA)^{-1}A^T$. Derive $A^\dagger$ in terms of U, V, and $\Sigma^\dagger$.

   (c) Use the result in (b) to implement $\mathbf{x} = \sum_{i=1}^{n} \frac{\mathbf{u}_i^T\mathbf{b}}{\sigma_i}\mathbf{v}_i$, where $\mathbf{u}_i$ and $\mathbf{v}_i$ being the i-th column of U and V respectively.

3. (20%) Consider the one-dimensional diffusion equation

$$\begin{cases} u_t = Du_{xx} + f(x,t), & 0 < x < 1, \quad t > 0, \\ u(x,0) = u_0(x), & 0 \leq x \leq 1, \\ u(0,t) = g(t), \quad u(1,t) = h(t), & t \geq 0, \end{cases}$$

where $u = u(x,t)$, $D > 0$ is a diffusivity constant, and $f(x,t)$ is a source term.

   (a) Denoting the mesh size and time step by $h = 1/m$ (for some $m \in \mathbb{N}$) and Δt respectively. Develop an *explicit* numerical scheme based on finite difference method (first-order in time, second-order in space) for the diffusion equation.

   (b) Derive the local truncation error of your proposed the scheme in (a).

   (c) Is there any constraint on the time step size of the scheme in (a)? Explain in detail.

4. (20%) For any initial guess $\mathbf{x}^{(0)} \in \mathbb{R}^n$, the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ is defined by $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$ for $k \geq 1$, where $T \in \mathbb{R}^{n\times n}$ and $\mathbf{c} \in \mathbb{R}^n$. Prove that the sequence converges to the unique solution $\mathbf{x} = T\mathbf{x} + \mathbf{c}$ if and only if the spectral radius $\rho(T) < 1$.

5. (20%) The initial value problem is given as follows:

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t} = f(t, y(t)), \quad a < t < b, \\ y(a) = \alpha. \end{cases}$$

Let the step size $\Delta t = \dfrac{b-a}{N}$ and $t_i = a + i\Delta t$ for $i = 0, 1, \cdots, N$. Denote the solution approximation $w_i \approx y(t_i)$, the *second-order midpoint Runge-Kutta* method for the IVP reads

$$\begin{cases} w_0 = \alpha, \\ w_{i+1} = w_i + \Delta t \cdot a_1 f(t + \alpha_1, w_i + \beta_1), \quad \text{for} \quad i = 0, 1, \cdots, N-1. \end{cases}$$

Derive the coefficients a_1, α_1, and β_1.

6. (10%) Assume that $A \in \mathbb{R}^{n\times n}$ has n distinct eigenvalues with an associated collection of linearly independent eigenvectors $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \cdots, \mathbf{v}^{(n)}\}$. Let λ_1 be the dominant eigenvalue, prove that there exist a nonzero vector $\mathbf{x}^{(0)} \in \mathbb{R}^n$ such that the sequence

$$\mathbf{x}^{(1)} = A\mathbf{x}^{(0)}, \quad \mathbf{x}^{(2)} = A\mathbf{x}^{(1)}, \quad \mathbf{x}^{(k)} = A\mathbf{x}^{(k-1)}$$

converges to the dominant eigenvector of A.