Qualifying Exam for Numerical Analysis
Department of Mathematics, National Central University
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1. (10%) Let $f \in C^2[a,b]$ and $p \in (a,b)$ such that $f(p) = 0$ and $f'(p) \neq 0$. Assume that the sequence generated by Newton's method

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$$

converges to p as $n \to \infty$. Prove that this sequence converges to p *quadratically*.

2. (20%) Consider a linear system $A\mathbf{x} = \mathbf{y}$ where A is an invertible tridiagonal matrix of the form

$$A = \begin{bmatrix} a_1 & c_1 & & & \\ b_2 & a_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & b_{m-1} & a_{m-1} & c_{m-1} \\ & & & b_m & a_m \end{bmatrix} \in \mathbb{R}^{m\times m} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{m-1} \\ y_m \end{bmatrix} \in \mathbb{R}^m.$$

Implement the Thomas's algorithm via the following two steps.

   (a) Applying Gaussian elimination without pivoting yields the LU factorization $A = LU$, where

$$L = \begin{bmatrix} 1 & & & & \\ \beta_2 & 1 & & & \\ & \ddots & \ddots & & \\ & & \beta_{m-1} & 1 & \\ & & & \beta_m & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} \alpha_1 & c_1 & & & \\ & \alpha_2 & c_2 & & \\ & & \ddots & \ddots & \\ & & & \alpha_{m-1} & c_{m-1} \\ & & & & \alpha_m \end{bmatrix}.$$

   Derive the coefficients α_j's (for $j = 1, 2, \cdots, m$) and β_j's (for $j = 2, 3, \cdots, m$).

   (b) Use the above LU factorization to find the solution $\mathbf{x}$ that is related to the coefficients α_j's and β_j's.

3. (10%) Given the dataset $\{(x_i, y_i)\}_{i=1}^m$, we aim to fit the data simply using the line $y = a_1 x + a_0$ via minimizing the *weighted least squares error*

$$E(a_0, a_1) = \sum_{i=1}^{m} w_i (y_i - a_1 x_i - a_0)^2,$$

where $w_i > 0$ are given weighted values for all $i = 1, 2, \cdots, m$. Derive the normal equation for solving $\mathbf{x} = [a_0, a_1]^T$.

4. (20%) Consider a rectangular linear system $A\mathbf{x} = \mathbf{b}$, where $A \in \mathbb{R}^{m\times n}$ with $m > n$ and $\mathbf{b} \in \mathbb{R}^m$. Suppose A is full rank ($\text{rank}(A) = n$). Derive the solution $\mathbf{x}$ in the sense that $\mathbf{x}$ minimizes the least squares error $\|\mathbf{b} - A\mathbf{x}\|_2$ using singular value decomposition (SVD) $A = U\Sigma V^T$, where $U \in \mathbb{R}^{m\times m}$, $\Sigma \in \mathbb{R}^{m\times n}$, and $V \in \mathbb{R}^{n\times n}$.

5. (20%) The two-point boundary value problem is given as follows:

$$\begin{cases} u''(x) = f(x), \quad a < x < b, \\ u(a) = \alpha, u(b) = \beta. \end{cases}$$

Let the mesh size $h = \dfrac{b-a}{m}$ for some $m \in \mathbb{N}$ and the grid point $x_j = a + jh$ for $j = 0, 1, \cdots, m$.

(a) Use the method of undetermined coefficients to find the finite difference formula for $u''(x_j)$ with the given nodal values $u(x_{j-1})$, $u(x_j)$,and $u(x_{j+1})$. Here we suppose $u \in C^\infty(\mathbb{R})$.

(b) Using the finite difference formula implemented in (a) results in the linear system $A\mathbf{U} = \mathbf{F}$ where $\mathbf{U} = [U_1, U_2, \cdots, U_{m-1}]^T$ denotes the numerical solution that approximates the exact solution $\hat{\mathbf{U}} = [u(x_1), u(x_2), \cdots, u(x_{m-1})]^T$. Derive the matrix A and right-hand side vector $\mathbf{F}$.

(c) Suppose $\|A^{-1}\| \leq C$ and the constant C is independent of m (or the mesh size h), prove that $\mathbf{U}$ converges to $\hat{\mathbf{U}}$ quadratically when $h \to 0$.

6. (20%) Consider the one-dimensional diffusion equation

$$\begin{cases} u_t = Du_{xx}, & 0 < x < 1, \quad t > 0, \\ u(x,0) = f(x), & 0 \leq x \leq 1, \\ u(0,t) = g(t), \quad u(1,t) = h(t), & t \geq 0, \end{cases}$$

where $u = u(x,t)$ and $D > 0$ is a diffusivity constant.

(a) Develop an *explicit* numerical scheme based on finite difference method (first-order in time, second-order in space) for the diffusion equation.

(b) Derive the local truncation error of your proposed the scheme in (a).

(c) Is there any constraint on the time step size of the scheme in (a)? Explain in detail.