Department of Mathematics, National Central University, Qualifying Exam for Numerical Analysis

Fall 2025

Instructions: Do all 5 problems. Show all work. State clearly any theorems you use.

1. Consider Newton's method and the Secant method for solving the nonlinear equation

$$f(x) = 0, \qquad f : \mathbb{R} \to \mathbb{R}.$$

   (a) (5 pts) Describe the *geometric interpretation* of Newton's method and the Secant method for finding the roots of $f(x) = 0$. Illustrate how each method generates successive approximations of a root.

   (b) (5 pts) Give some examples of situations where Newton's method fails to converge to the desired solution. Explain the underlying reasons for failure and provide some resolution to avoid such difficulty.

   (c) (5 pts) Suppose you are given the polynomial

$$p(x) = (x-1)(x-2)(x-3),$$

   but you do not know the locations of the roots in advance. Propose a systematic procedure for finding *all* the roots of $p(x)$ using Newton's method. Justify why your procedure is reasonable.

2. Consider the linear system $A\boldsymbol{x} = \boldsymbol{y}$, where $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times m}$ is a *symmetric*, tridiagonal nonsingular matrix

$$A = \begin{bmatrix} a_1 & c_1 & & & \\ c_1 & a_2 & c_2 & & \\ & c_2 & a_3 & \ddots & \\ & & \ddots & \ddots & c_{m-1} \\ & & & c_{m-1} & a_m \end{bmatrix}.$$

   (a) (5 pts) Show that performing Gaussian elimination without pivoting yields the factorization

$$A = LDL^\top,$$

   where L is unit lower bidiagonal and $D = \operatorname{diag}(d_1, \ldots, d_m)$. Derive the recurrences

$$d_1 = a_1, \qquad \ell_j = \frac{c_{j-1}}{d_{j-1}}, \qquad d_j = a_j - l_j\, c_{j-1}, \quad j = 2, \ldots, m,$$

   and write the factors explicitly as

$$L = \begin{bmatrix} 1 & & & \\ \ell_2 & 1 & & \\ & \ddots & \ddots & \\ & & \ell_m & 1 \end{bmatrix}, \qquad D = \operatorname{diag}(d_1, \ldots, d_m).$$

(b) (10 pts) Solve $A\boldsymbol{x} = \boldsymbol{y}$ using the factorization using $A = LDL^\top$. Write down the algorithm and show the overall of algorithm is is an $O(m)$ algorithm.

(c) (5 pts) Assume A is symmetric positive definite (SPD). Prove that $d_j > 0$ for all $j = 1, \ldots, m$. Conclude that the factorization is well-defined without pivoting and that the algorithm in part (a) is numerically stable for SPD matrices.

3. Consider the matrix

$$A = \begin{pmatrix} 3 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 8 & -0.5 \\ 0 & 0 & -0.5 & 11 \end{pmatrix}.$$

(a) (5 pts) Sketch the Gershgorin disks for the matrix A and give a bound for the spectral radius $\rho(A)$.

(b) (7 pts) Derive an algorithm using a variant of power method to compute the second smallest eigenvalue of the matrix. Justify your approach.

(c) (5 pts) Estimate the convergence rate of the power method for finding the largest eigenvalue in magnitude, and propose an approach to accelerate its convergence.

(d) (8 pts) Show that the sequence generated by the Jacobi method for solving $Ax = b$ converges for any initial guess.

Hint: Write down the Jacobi iteration matrix $T_J = I - D^{-1}A$. Recall that similar matrices have the same eigenvalues. Consider applying a diagonal similarity transform and then use Gershgorin's theorem to argue that all eigenvalues of T_J lie strictly inside the unit disk.

4. Let $f : [a, b] \to \mathbb{R}$ be twice continuously differentiable ($f \in C^2([a, b])$).

(a) (10 pts) Let $x_0 = a$, $x_1 = b$, and $h = b - a$. Using the **linear Lagrange interpolant** through the nodes a and b, derive the trapezoid rule for $\int_a^b f(x)\,dx$ together with an error term.

(b) (10 pts) Let $a = x_0 < x_1 < \cdots < x_n = b$ be a uniform partition of $[a, b]$ with mesh size $h = (b - a)/n$. Derive the **composite trapezoid rule** for approximating

$$\int_a^b f(x)\,dx$$

and prove that its error is

$$-\frac{1}{12}(b - a)h^2 f''(\xi), \qquad \text{for some } \xi \in (a, b).$$

5. Consider the following initial-value problem

$$\begin{cases} x'(t) = -10x \\ x(0) = 1 \end{cases}$$

(a) (10 pts) We employ the **second-order Runge-Kutta method (RK2)**,

$$x(t+h) = x(t) + \frac{1}{2}(K_1 + K_2)$$

where

$$\begin{aligned} K_1 &= hf(t,x) \\ K_2 &= hf(t+h, x+K_1) \end{aligned}$$

for solving the above initial value problem. Find the condition on the grid size h so that the RK2 method is stable for this problem? Justify your answer.

(b) (10 pts) Write down the forward Euler method and what the bound on the step size h such that the forward Euler method stable?