Qualifying Exam for Numerical Analysis
Department of Mathematics, National Central University
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1. (15%) Consider the m-by-m matrix

$$A = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ -1 & -1 & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ -1 & -1 & \cdots & -1 & 1 \end{bmatrix}_{m\times m}.$$

   (a) Derive the inverse matrix of A.

   (b) Find the condition number $K(A) = \|A\|\|A^{-1}\|$ in infinity norm.

   (c) Suppose $\tilde{\mathbf{x}}$ is the approximation solution to the linear system $A\mathbf{x} = \mathbf{b}$ with the residual vector $\mathbf{r} = \mathbf{b} - A\tilde{\mathbf{x}}$. If $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$, show taht

$$\frac{\|\mathbf{x} - \tilde{\mathbf{x}}\|}{\|\mathbf{x}\|} \leq K(A)\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|}.$$

2. (15%) Use the method of undetermined coefficients to find the finite difference formula for $u''(\bar{x})$ with the given nodal values $u(\bar{x} - h)$, $u(\bar{x})$, and $u(\bar{x} + h)$. Here we suppose $u \in C^\infty(\mathbb{R})$.

   (a) Write down the finite difference formula for $u''(\bar{x})$, $D^2u(\bar{x})$, with your implementation carefully.

   (b) What is the local truncation error for $D^2u(\bar{x})$? Do NOT answer by big O notation only.

   (c) Find the optimal value of h so that the most accurate approximation can be achieved with round-off error ε in a machine.

3. (20%) The initial value problem is given as follows:

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t} = f(t, y(t)), \quad a < t < b, \\ y(a) = \alpha. \end{cases}$$

   Let the step size $\Delta t = \dfrac{b-a}{N}$ and $t_i = a + i\Delta t$ for $i = 0, 1, \cdots, N$. Denote the solution approximation $w_i \approx y(t_i)$, the *second-order midpoint Runge-Kutta* method for the IVP reads

$$\begin{cases} w_0 = \alpha, \\ w_{i+1} = w_i + \Delta t \cdot a_1 f(t + \alpha_1, w_i + \beta_1), \quad \text{for} \quad i = 0, 1, \cdots, N-1. \end{cases}$$

   Derive the coefficients a_1, α_1, and β_1.

4. (15%) The quadratic function is defined by $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T Q\mathbf{x}$, where Q is a $n\times n$, real, symmetric positive matrix.

   (a) Compute $\nabla f(\mathbf{x})$ and write down the gradient descent method with the step size μ.

   (b) Find the constraint for μ so that the gradient descent iteration is guaranteed to be convergent.

5. (15%) Consider the linear system $A\mathbf{x} = \mathbf{b}$, where A is the tridiagonal matrix given by

$$A = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}_{m\times m}.$$

It is known that the set $V = \{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \cdots, \mathbf{v}^{(m)}\}$ with the component

$$\mathbf{v}_j^{(p)} = \sin\left(\frac{pj\pi}{m+1}\right)$$

for $p = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, m$ gives the eigenvector with the corresponding eigenvalue

$$\lambda_p = 4\sin^2\left(\frac{p\pi}{2(m+1)}\right)$$

of A.

(a) Write down the Jacobi iteration for the linear system.

(b) Show that the iteration method in (a) must converge to the solution with arbitrary initial guess.

(c) Develop a direct solver using the orthogonality of V. [Hint: Let $\mathbf{x} = \sum_{p=1}^{m} x_p \mathbf{v}^{(p)}$ and determine x_p.]

6. (20%) Consider the one-dimensional diffusion equation

$$\begin{cases} u_t = Du_{xx}, & 0 < x < 1, \quad t > 0, \\ u(x,0) = f(x), & 0 \le x \le 1, \\ u(0,t) = g(t), \quad u(1,t) = h(t), & t \ge 0, \end{cases}$$

where $u = u(x,t)$ and $D > 0$ is a diffusivity constant.

(a) Develop an *implicit* numerical scheme based on finite difference method (first-order in time, second-order in space) for the diffusion equation.

(b) Derive the local truncation error of your proposed the scheme in (a).

(c) Is there any constraint on the time step size of the scheme in (a)? Explain in detail.