P.h.D. Qualify Examination: Graph theory

1. State the following theorems:
   (a) (5%) Graph theoretic formulation for Hall's marriage theorem
   (b) (5%) Ore's theorem on Hamiltonian cycles
   (c) (5%) Wagner's theorem on forbidden graph characterization of finite planar graphs
2. (10%) Let T be a tree with k edges. Suppose that G is a simple graph whose minimum degree satisfies $\delta(G) \geq k$. Show that G contains T as a subgraph.
3. (15%) Prove that a finite simple graph G of order $n \geq 2$ with at least
$$\frac{(n-1)(n-2)}{2} + 1$$
edges must be connected.
4. (15%) Let G be a finite simple graph of order $n \geq 3$ having at least
$$\frac{(n-1)(n-2)}{2} + 2$$
edges. Prove that G has a Hamilton cycle.
5. Let G denote a finite graph without loops. Given any positive integer k, let $P_G(k)$ denote the number of proper colorings of G with k colors.
   (a) (5%) Given an edge e of G, let $G - e$ and G/e denote the graphs obtained from G by deleting e and contracting e, respectively. Prove that
$$P_G(k) = P_{G-e}(k) - P_{G/e}(k)$$
   for any positive integer k and any edge e of G.
   (b) (5%) Prove that if G is of order n, then $P_G(x)$ is a monic polynomial in x of degree n.
   (c) (5%) Prove that G is connected if and only if the coefficient of x in $P_G(x)$ is nonzero.
   (d) (10%) Prove that if G is of order n, then $P_G(x)$ has no real root greater than $n - 1$.
6. Let $(d_1, d_2, \ldots, d_n)$ be a nonincreasing sequence of positive integers with length $n \geq 2$.
   (a) (10%) Prove that there exists a tree of order n with the degree sequence $(d_1, d_2, \ldots, d_n)$ if and only if $d_1 + d_2 + \cdots + d_n = 2(n-1)$.
   (b) (5%) Design an algorithm with the input $(d_1, d_2, \ldots, d_n)$ that either outputs a tree with the degree sequence $(d_1, d_2, \ldots, d_n)$ or returns no tree with the degree sequence $(d_1, d_2, \ldots, d_n)$.
   (c) (5%) Prove the correctness of the algorithm that you give in the part (b).