National Central University
PhD. Qualify Examination for Graph theory

1. (a) (5%) State the Matrix-Tree theorem.
   (b) (10%) Prove that there are n^{n-2} trees with vertex set $\{1, 2, \ldots, n\}$.

2. (15%) Let $(d_1, d_2, \ldots, d_n)$ be a nonincreasing sequence of positive integers with length $n \geq 2$. Prove that there exists a tree of order n with the degree sequence $(d_1, d_2, \ldots, d_n)$ if and only if
$$d_1 + d_2 + \cdots + d_n = 2(n-1).$$

3. (15%) Let $n \geq 1$ denote an integer. Prove that if a n-vertex simple graph G has degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$, then the chromatic number $\chi(G)$ of G satisfying
$$\chi(G) \leq 1 + \max_{1 \leq i \leq n} \min\{d_i, i-1\}.$$

4. (15%) Let G denote a finite simple graph of order $n \geq 3$. Prove that G is 2-connected if and only if for every two vertices x, y of G there is a cycle containing both.

5. (15%) Let G be a finite simple graph of order $n \geq 3$ having at least
$$\frac{(n-1)(n-2)}{2} + 2$$
edges. Prove that G has a Hamilton cycle.

6. Let G denote a finite simple graph. Suppose that G is a planar graph with n vertices, e edges and f faces.
   (a) (5%) Prove that $n - e + f = 2$.
   (b) (5%) Prove that if $n \geq 3$ then $e \leq 3n - 6$.
   (c) (5%) Prove that there exists a vertex v of G with degree ≤ 5.
   (d) (10%) Prove that G is 5-colorable.