圖論

(1) (15%) ① If G is k-critical (i.e., $\chi(G)=k$, $\chi(G-x)=k-1$ for every $x \in V(G)$ where $\chi(G)$ is the chromatic number of a graph G), show that the minimum degree $\delta(G) \geq k-1$.

② If $\chi(G)=k$, show that G contains at least k vertices each of which has degree at least $k-1$.

(2) (10%) A graph G is 3-edge connected. Show that every edge of G is the intersection of two cycles in G.

(3) (10%) Show that if $\operatorname{diam} G \geq 3$ ($\operatorname{diam} G$ is the diameter of G) then $\operatorname{diam} G^c \leq 3$.

(4) (20%) ① Show that every tournament has a directed Hamiltonian path.

② Show that every tournament contains a vertex from which every vertex is reachable by a directed path of length at most 2.

5 (25%) ① State the edge version of Menger's Theorem for directed graph.

② State and prove Hall's Theorem (i.e., matching theorem, marriage theorem, SDR theorem).

③ Prove that in a bipartite graph G, the maximum number of edges in a matching of G is equal to the minimum number of vertices needed to cover all the edges in G

④ Show that every k-regular bipartite graph ($k \geq 1$) is 1-factorizable (i.e., contains a perfect matching).

6. (15%) G is a graph of order $n \geq 3$.

① Show that if $\deg x + \deg y \geq n$ for every pair of nonadjacent vertices x, y of G, then G contains a Hamiltonian cycle.

② Show that if $\deg x + \deg y \geq n-1$ for every nonadjacent vertices x, y of G, then G contains a Hamiltonian path.

7. Show that the petersen's graph (5%)



is not planar.