NCU Ph.D Qulification Exam. Differential Geometry

August 21 2025

State your assumptions of cite a known theorem if you are using it, Present your solutions as simple as possible yet rigorous enough.

1. (10 pts) Let S^2 be the unit sphere $x^2+y^2+z^2=1$ in $\mathbb{R}^3$. Define in S^2 the six charts corresponding to the six hemispheres–the front, rear, right, left, upper, and lower hemispheres:

$$\begin{aligned} U_1 &= \{x,y,z) \in S^2 \mid x>0\}, \quad \phi_1(x,y,z) = (y,z), \\ U_2 &= \{x,y,z) \in S^2 \mid x<0\}, \quad \phi_2(x,y,z) = (y,z), \\ U_3 &= \{x,y,z) \in S^2 \mid y>0\}, \quad \phi_3(x,y,z) = (x,z), \\ U_4 &= \{x,y,z) \in S^2 \mid y<0\}, \quad \phi_4(x,y,z) = (x,z), \\ U_5 &= \{x,y,z) \in S^2 \mid z>0\}, \quad \phi_5(x,y,z) = (x,y), \\ U_6 &= \{x,y,z) \in S^2 \mid z<0\}, \quad \phi_6(x,y,z) = (x,y). \end{aligned}$$

   Describe the domain $\phi_6(U_{16})$ of $\phi_1 \circ \phi_6^{-1}$ and show that $\phi_1 \circ \phi_6^{-1}$ is C^∞ on $\phi_6(U_{16})$. [Notation: $U_{16} := U_1 \cap U_6$.]

2. (15 pts) The unit sphere S^n in $\mathbb{R}^{n+1} = \{(x_1, x_2, \ldots, x_{n+1}) \mid x_i \in \mathbb{R}\}$ is defined by the equation $\sum_{i=1}^{n+1}(x_i)^2 = 1$. For $p = (p_1, \ldots, p_{n+1}) \in S^n$, show that a necessary and sufficient condition for $X_p = \sum_{j=1}^{n+1} a_j \partial/\partial x_j|_p \in T_p(\mathbb{R}^{n+1})$ to be tangent to S^n at p is $\sum_{i=1}^{n+1} a_i p_i = 0$.

3. (10 pts) Consider the C^∞ map $F : \mathbb{R} \to \mathbb{R}^2$, $F(t) = (t^2, t^3)$.

   (a) Is F a one-to-one map? Verify your answer.

   (b) Is F_* injective? Verify your answer.

   (c) Is F an immersion? Why?

4. (15+10 pts) Let $X = \frac{\partial}{\partial y} + z\frac{\partial}{\partial x}$, $Y = x\frac{\partial}{\partial z} + y\frac{\partial}{\partial x}$ be vector fields in $\mathbb{R}^3$, $\omega = xdy\wedge dz + zdx\wedge dy \in \Omega^2(\mathbb{R}^3)$.

   (a) Find $d\omega$, $\iota_X\omega$, $\mathcal{L}_X\omega$.

   (b) Find $\mathcal{L}_X Y$.

5. (20 pts) Consider the Riemannian manifold (M, g) where $M = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}$ and $g = \frac{1}{1-x^2-y^2}\Big((1+x^2)dx\otimes dx + xy(dx\otimes dy + dy\otimes dx) + (1+y^2)dy\otimes dy\Big)$.

   (a) Find an *orthonormal frame* E_1, E_2 on (M, g) (i.e., (E_1, E_2) form an orthonormal basis for T_pM at each $p \in M$).

   (b) Find grad$f$ where f is the C^∞ function on M given by $f(x,y) = x^2 - y^2$. Express grad$f$ in terms of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

6. (20 pts) Let $\gamma(t) = (a(t), b(t))$, $t \in I$ (an open interval), be a smooth injective curve in the xz-plane, and suppose $a(t) > 0$, and $\dot\gamma(t) \neq 0$ for all $t \in I$. Let $M \subset \mathbb{R}^3$ be the surface of revolution obtained by revolving the image of γ about the z-axis. The map $\phi(\theta, t) = (a(t)\cos\theta, a(t)\sin\theta, b(t))$ from $\mathbb{R}\times I$ to $\mathbb{R}^3$ is a local parametrization of M in a neighborhood of any point. Assume that the curve γ is unit speed.

   (a) Compute the expression for the induced metric on M in (θ, t) coordinates.

   (b) Compute the Christoffel symbols of the induced metric in (θ, t) coordinates.

   (c) Show that each "meridian" $\{\theta = \theta_0\}$ is a geodesic on M.

   (d) Determine necessary and sufficient conditions for a "latitude circle" $\{t = t_0\}$ to be a geodesic.