Qualifying Exam - Differential Equations

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2025 Fall

The following Problems are independent from one to each others.

Problem 1 (25pts)**.** Consider the initial value problem of the unknown function x,

$$\begin{cases} \frac{dx}{dt} = f(t,x), \\ x(t_0) = x_0. \end{cases} \tag{I.V.P.}$$

1.) State the theorem of (local) existence and uniqueness of the solution to (I.V.P.). Also, you should give the appropriate assumptions on f, t_0, x_0.

2.) Prove this theorem. You can follow the instructions.

- Transform (I.V.P.) into a integral equation.
- Introduce the Picard's iterations.
- Find out the suitable space of functions.
- Prove the convergence.

Problem 2 (15pts)**.** Let K be a real-valued continuous function on the square $[a,b]\times[a,b] \subset \mathbb{R}^2$. We suppose that for all $(x,y) \in [a,b]\times[a,b]$, $0 \leq K(x,y) \leq M$ with $M(b-a) \leq \frac{1}{2}$. Introduce the constant functions $u_0 = 0$ and $v_0 = 2$ on $[a,b]$ as well as the iterations,

$$\forall n \in \mathbb{N},\ x \in [a,b], \quad u_{n+1}(x) = \int_a^b K(x,y)u_n(y)dy + 1,$$
$$v_{n+1}(x) = \int_a^b K(x,y)v_n(y)dy + 1.$$

Show that the sequence of functions $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ converge both uniformly on $[a,b]$ to the function φ which satisfies the integral equation,

$$\varphi(x) = \int_a^b K(x,y)\varphi(y)dy + 1.$$

Also, show that the inequalities hold,

$$\forall x \in [a,b], \quad u_0(x) \leq u_1(x) \leq \cdots \leq \varphi(x) \leq \cdots \leq v_1(x) \leq v_0(x).$$

Problem 3 (25pts)**.** Let $A \in \mathcal{M}_n(\mathbb{C})$ be a complex-valued $n \times n$ matrix. Consider a linear system of differential equations with constant coefficients,

$$\frac{dx}{dt} = Ax.$$

Show that the solutions to this system are linear combinations of the functions $x_1, \cdots, x_n$, which can be written as

$$\forall 1 \leq j \leq n, \quad x_j(t) = e^{\lambda_j t} p_j(t),$$

where $\lambda_j \in \mathbb{C}$ and p_j are polynomial functions. You should give all λ_j and p_j in terms of A.

Problem 4 (20pts)**.** Let a be a continuous and bounded function on $(0, +\infty)$. Consider the following differential equation posed on $(0, +\infty)$,

$$\frac{d^2x}{dt^2}(t) + a(t)x(t) = 0.$$

We assume that this equation possesses a solution φ which tends to 0 at $+\infty$. Show that there exists another solution ψ which is unbounded on $(0, +\infty)$.

Problem 5 (15pts)**.** Consider the system of differential equations of unknown real-valued functions x_1 and x_2,

$$\begin{cases} \frac{dx_1}{dt} &= -x_2 + x_1(1 - x_1^2 - x_2^2), \\ \frac{dx_2}{dt} &= x_1 + x_2(1 - x_1^2 - x_2^2). \end{cases}$$

Justify that $t \mapsto (\cos t, \sin t)$ is a solution to this system. Then study the orbital stability of this solution.

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