Qualify Exam

Subject: Differential Equations

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1. (17 points) Suppose that $g \in C^1(\mathbb{R})$, and $f, k \in C(\mathbb{R})$. Please show that the solution of the initial value problem

$$\begin{cases} y'' + f(y)y' + g(y) = \int_0^t k(s)y(s)\ ds, \\ y(0) = 1, \\ y'(0) = 2. \end{cases}$$

exists locally and is unique. Please show your work in detail.

2. (17 points) Suppose y is a solution of

$$\begin{cases} y' = f(t)\sin y + \varepsilon\cos(g(t,y)), \\ y(t_0) = y_0, \end{cases}$$

and z is a solution of

$$\begin{cases} z' = f(t)\sin z + \varepsilon\cos(h(t,z)), \\ z(t_0) = z_0, \end{cases}$$

where f, g, h are continuous functions and ε, y_0, z_0 are constants. Give an estimate of $|y(t) - z(t)|$ for any $t > t_0$. Please show your work in detail.

3. (15 points) Consider the initial value problem

$$\begin{cases} x'' + \left(\frac{x^2}{1+x^2}\right)x' + x^3 = 0, \\ x(0) = x_0, \\ x'(0) = y_0, \end{cases}$$

where x_0, y_0 are two constants. Please show that

$$\lim_{t\to\infty} x(t) = 0, \quad \text{and} \quad \lim_{t\to\infty} x'(t) = 0.$$

Hint. Use reduction of order to the equation, and construct a Lyapunov function for the system.

4. (A) (10 points) Does it exist a nontrivial periodic solution in region $x^2 + y^2 < 4$ for the system

$$\begin{cases} x' = 8x - xy^2 - x^3y^2 \\ y' = 4y - x^2y + \frac{1}{3}x^2y^3 \end{cases} ?$$

(B) (10 points) Show that the system

$$\begin{cases} x' = 2(x - y) - x^3 \\ y' = 2(x + y) - y^3 \end{cases}$$

has a periodic orbit in some annulus region centered at the origin.

5. (15 points) Determine the stability of equilibria of the system

$$\begin{cases} x' = 2x - \frac{1}{2}x^2 - xy, \\ y' = -3y - y^2 + xy. \end{cases}$$

Find the stable and unstable subspaces of the linearized system at each equilibrium.

6. (16 points) Consider the Riccati equation

$$y' + g(x)y + h(x)y^2 = f(x) \qquad (*)$$

where $g, f \in C^0(\mathbb{R})$, $h \in C^1(\mathbb{R})$, and $h(x) \neq 0$ in $\mathbb{R}$.

(A) Show equation $(*)$ can be transformed into a second order linear equation of u, where $u(x) \equiv e^{\int^x h(s)y(s)ds}$. Find that equation of u.

Hint. $u' = hyu$

(B) Use the result in (A) to construct a particular solution of

$$y' - y - e^x y^2 = 2e^{-x}.$$