Qualifying Exam.: Ordinary Differential Equations August 26, 2022

1.(25%) Consider the initial value problem:

$$\frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0, \tag{1}$$

where f is a function from $D \subseteq \mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, and D is an open set containing (t_0, x_0). Please state and prove the existence and uniqueness of solutions of the initial value problem.

2. (15%) Consider the system of differential equations:

$$\frac{dx}{dt} = A(t)x, \quad -\infty < t < \infty, \tag{2}$$

where $x(t) \in \mathbb{R}^n$ and $A(t) = A(t + T)$ is a continuous $n \times n$ periodic matrix for some $T > 0$. Let $\Phi(t)$ be a fundamental matrix for (2), show that there exists a nonsingular T-periodic matrix $P(t)$ and a constant matrix R such that $\Phi(t) = P(t)e^{tR}$.

3. (20%) (1) Please state the stable manifold theoerm for the nonlinear system:

$$\frac{dx}{dt} = f(x). \tag{3}$$

(2) Find the general solutions of the system

$$\begin{cases} \dfrac{dx_1}{dt} = -x_1, \\ \dfrac{dx_2}{dt} = -x_2 + x_1^2, \\ \dfrac{dx_3}{dt} = x_3 + x_1^2, \end{cases}$$

and determine the stable and unstable manifolds for the system.

4. (20%) (1) Determine the stability property of the critical point at the origin for the following system:

$$\begin{cases} y_1' = y_1^3 - y_2^3, \\ y_2' = 2y_1y_2^2 + 4y_1^2y_2 + 2y_2^3. \end{cases}$$

(2) Prove that the second order differential equation

$$x'' + (x^2 + 2(x')^2 - 1)x' + x = 0$$

has a non-trivial periodic solution.

5. (20%) Consider the system:

$$\begin{cases} \dfrac{dx_1}{dt} = ax_1 - bx_1x_2 - ex_1^2, \\ \dfrac{dx_2}{dt} = -cx_2 + dx_1x_2 - fx_2^2, \end{cases} \tag{4}$$

where a, b, c, d, e, f are positive constants.

(1) Show that $\operatorname{div}(\varphi g(x_1, x_2)) \neq 0$ for $x_1 > 0, x_2 > 0$, where $g(x_1, x_2) = (ax_1 - bx_1x_2 - ex_1^2, -cx_2 + dx_1x_2 - fx_2^2)$ and $\varphi = 1/(x_1x_2)$.

(2) Show that system (4) has no closed orbits in the first quadrant.