Ph. D. Qualifying Examination, 2003

Analysis

歸檔

Choose any 5 problems in the following test.

1.(a) Prove that every function of bounded variation has at most a countable number of discontinuities.

(b) Let $f(x) = x\sin(1/x)$ for $0 < x \leq 1$ and $f(0) = 0$. Show that f is bounded and continuous on $[0,1]$, but the variation of $f, V[f;0,1] = +\infty$.

2. For $1 \leq p < \infty$, we define the spaces $(l^p, ||\cdot||_p)$ and $(l^\infty, ||\cdot||_\infty)$ as the following respectively

$$l^p = \{< x_i >_{i=1}^\infty : \ (|| < x_i > ||_p)^p = \sum_{i=1}^\infty |x_i|^p < \infty, x_i \in R \ \forall i\},$$

$$l^\infty = \{< x_i >_{i=1}^\infty : \ || < x_i > ||_\infty = \sup|x_i| < \infty, x_i \in R \ \forall i\}.$$

Prove that l^p is complete and l^∞ is a Banach space.

3. Let v_n be the volume of the unit ball in R^n. Show by using Fubini's theorem that

$$v_n = 2v_{n-1}\int_0^1 (1-t^2)^{\frac{n-1}{2}}\, dt.$$

4. Prove Egoroff's Theorem : If $< f_n >$ is a sequence of measurable functions that converge to a real-values function f a.e. on a measurable set E of finite measure, then given $\eta > 0$, there is a subset $A \subset E$ with $m(A) < \eta$ such that f_n converges to f uniformly on $E\backslash A$.

5. Let $1 \leq r,\ p_1, p_2, \cdots, p_k \leq \infty$ and $\frac{1}{p_1} + \cdots + \frac{1}{p_k} = \frac{1}{r}$. Prove that : if $f_1 \in L^{p_1}, \cdots, f_k \in L^{p_k}$, then

$$||f_1 \cdots f_k||_r \leq ||f_1||_{p_1} \cdot ||f_2||_{p_2} \cdots ||f_k||_{p_k}.$$

6.(a) Let $K \in L^1(R^n)$, $\int_{R^n} K = 1$ and $K_\epsilon(x) = \epsilon^{-n} K(\frac{x}{\epsilon})$. Define

$$f_\epsilon(x) = (f * K_\epsilon)(x) = \int_{R^n} f(t) K_\epsilon(x - t)\, dt, x \in R^n.$$

Prove that if $f \in L^p(R^n), 1 \leq p < \infty$, then $||f_\epsilon - f||_p \to 0$ as $\epsilon \to 0$.

(b) Prove that C_0^∞ is dense in $L^p(R^n)$ for $1 \leq p < \infty$.

7.(a) Let f be a nonnegative function which is integrable over a measurable E. Prove that given any $\epsilon > 0$ there is $\delta > 0$ such that for every set $A \subset E$ with $m(A) < \delta$ we have

$$\int_A f < \epsilon.$$

(b) Let f be a Lebesgue integrable function on $[a, b]$, and

$$F(x) = F(a) + \int_a^x f(t)dt.$$

Prove that F is absolute continuous on $[a, b]$ and $F'(x) = f(x)$ for almost all x in $[a, b]$.