Department of Mathematics
National Central University

Ph. D. Qualifying Examination

Spring, 2001.

Analysis

Answer all of the following questions.

1. Suppose σ is a one-to-one transformation of the set of positive integers onto itself and a_n is a non-negative real number for all $n = 1, 2, \ldots$. Prove that
$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\sigma(n)}.$$

2. Let P be the class of all polynomials of the form
$$p(x) = \sum_{j=0}^{n} a_j x^{2j+1}.$$
Show that a continuous function f in $C[0,1]$ can be uniformly approximated on $[0,1]$ by a sequence in P if and only if $f(0) = 0$.

3. Suppose $f_n \in L^p(\mathbb{R})$, $\|f_n\|_p \leq M < \infty$, $n = 1, 2, \ldots$, where $1 < p < \infty$. Suppose f_n converges to f almost everywhere and that $g \in L^q(\mathbb{R})$, where $1/p + 1/q = 1$. Show that
$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)g(x)dx = \int_{\mathbb{R}} f(x)g(x)dx.$$

4. Let $f : [0,1] \longrightarrow \mathbb{R}$ be continuous and of bounded variation. Assume that for each $\epsilon > 0$, $f_{|[\epsilon,1]}$ is absolutely continuous. Show that f is absolutely continuous on $[0,1]$.

5. Let λ and μ be two positive Borel measures on $\mathbb{R}^n$ such that λ and μ are finite on compact sets and for every continuous function f on $\mathbb{R}^n$ with compact support,

$$\int_{\mathbb{R}^n} f d\lambda = \int_{\mathbb{R}^n} f d\mu.$$

Show that $\lambda = \mu$.

6. Let $f_n \to f$ on $[0,1]$ in the following sense: for every x in $[0,1]$, if $x_n \to x$, then $f_n(x_n) \to f(x)$. Show that f is continuous if all f_n are continuous.

7. Let $\{G_n\})_n$ be a sequence of non-empty open sets in $[0,1]$ with the Lebesgue measures $m(G_n) \leq 1/2^n$ for $n = 1, 2, \ldots$. Let

$$f(x) = \sum_{n=1}^{\infty} m(G_n \cap [0,1]), \quad 0 \leq x \leq 1.$$

Show that f is continuous, non-decreasing, and that $f'(x) = +\infty$ for all x in $\cap_{n=1}^{\infty} G_n$.

8. Prove or disprove: There exist continuous real-valued functions f and g defined on $[0,1]$ such that $f(x) = g(x)$ for uncountably many points x, but in every interval there exists a point x where $f(x) \neq g(x)$.