Algebra
Qualify Examination, Feb. 2025

All rings are assumed to have identity elements and all modules are unitary.

(10%) **1**. Classify all the groups of order 8.

(10%) **2**. Let $f : S_4 \to S_4$ be an automorphism of the symmetric group S_4 on four letters. Prove or disprove that there exists an element $\sigma \in S_4$ such that $f(x) = \sigma x \sigma^{-1}$, for any $x \in S_4$.

(10%) **3**. Let $p > 2$ be an odd prime. Prove or disprove that there are (up to isomorphism) exactly two distinct groups of order $2p$.

(20%) **4**. Let R be a commutative local ring. Prove or disprove that every finitely generated projective left R-module is free.

(20%) **5**. Prove or disprove that every (unitary) left module over a ring R with identity is isomorphic to a submodule of an injective left R-module.

(30%) **6**. Let L be an extension field of K.

(a) Let F be the set of all elements of L which are algebraic over K. Prove that F is a subfield of L.

(b) Suppose that L is a finite dimensional extension field of K. Prove that L is finitely generated and algebraic over K.

(c) Suppose that $K = (\mathbb{Q}, +, \cdot)$ is the field of rational numbers and L is a finite dimensional extension field of K. Prove that L is a simple extension, that is, $L = K(u)$ is the subfield generated by u over K, for some element $u \in L$.