Problem 1. (20%) In parts (a), (c) and (d), G is a finite group of odd order.

(a) Show that every conjugacy class of G contains an odd number of elements.

(b) Let H be an arbitrary group and let C be a conjugacy class in H. Suppose that, for some $h \in C$, we also have $h^{-1} \in C$. Show that for all $x \in C$, we also have $x^{-1} \in C$.

(c) Let $g \in G$ be an element not equal to the identity element. Show that g is not conjugate to g^{-1}.

(d) Let k be the number of conjugacy classes of G. Show that $k \equiv |G| \pmod 4$.

Problem 2. (15%) Let R be a commutative Noetherian ring with 1 and let M be a nonzero R-module. If $m \in M$, define the annihilator of m to be

$$\mathrm{Ann}(m) = \{r \in R \mid rm = 0\}.$$

(a) Suppose I is an ideal of R that is maximal among ideals that are annihilators of nonzero elements (that is, $I = \mathrm{Ann}(m)$ for some $0 \neq m \in M$, and if $I \subseteq \mathrm{Ann}(x)$ for some $0 \neq x \in M$, then $I = \mathrm{Ann}(x)$). Show that I is a prime ideal.

(b) Let $r \in R$ and $0 \neq m \in M$ be such that $rm = 0$. Show that there exists a prime ideal P of R and a nonzero element $x \in M$ such that $r \in P$ and $P = \mathrm{Ann}(x)$.

Problem 3. (15%) Let R be an integral domain.

(a) Suppose that every finitely generated R-module is free. Show that R is a field.

(b) Suppose that R is Noetherian and that every finitely generated R-module with no R-torsion is free. Show that R is a principal ideal domain.

Problem 4. (30%) Let L be a finite Galois extension of the field K and let $\alpha \in L$. Suppose there is an automorphism $\sigma \in \mathrm{Gal}(L/K)$ such that $\sigma(\alpha) = \alpha + 1$.

(a) Show that K has positive characteristic.

(b) If $L = K(\alpha)$, show that σ has order p, where p is the characteristic of K.

(c) Suppose σ generates $\mathrm{Gal}(L/K)$. Show that $\alpha^p - \alpha \in K$, where p is the characteristic of K.

Problem 5. (20%) Let R be a ring. Recall that an R-module P is *projective* if it satisfies the following condition: Whenever $f : A \to B$ is a surjective homomorphism of R-modules and $p : P \to B$ is a homomorphism, then there exists a homomorphism $h : P \to A$ such that $f \circ h = p$.

(a) Show that the direct sum of two projective modules is projective.

(b) Suppose we are given the following diagram of R-modules:

$$\begin{array}{ccccccccc} & & & & P_1 & & P_2 & & \\ & & & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow & & & & \downarrow & & \\ & & 0 & & & & 0 & & \end{array}$$

where the row is exact, P_1 and P_2 are projective, and α, γ are surjective.

Show that there exist a projective module P, homomorphisms t, s, and a surjection β such that the following diagram commutes and both rows are exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_1 & \xrightarrow{t} & P & \xrightarrow{s} & P_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

You should prove that the top row is exact, that the diagram commutes, and that β is surjective.