Algebra
Qualify Examination, Summer 2024

Let $(\mathbb{Z}, +, \cdot)$ and $(\mathbb{Q}, +, \cdot)$ denote the rings of integers and rational numbers, respectively. All rings are assumed to have identity elements and all modules are unitary.

(10%) **1**. Let n be a positive integer. Prove or disprove that there are only finitely many isomorphism classes of groups of order n (i.e., there are only finitely many groups of order n, up to isomorphism).

(10%) **2**. Let n be a positive integer. Suppose that G is a group of order pn such that $p > n$, for some prime number p. Prove or disprove that there is a unique subgroup of G of order p.

(10%) **3**. Let R be a Euclidean domain. Prove or disprove that the polynomial ring $R[x]$ in one variable x is Noetherian.

(10%) **4**. Let R be an integral domain. Prove or disprove that R is Artinian if and only if R is a field.

(10%) **5**. We view $(\mathbb{Q}, +)$ as a $\mathbb{Z}$-module. Prove or disprove that $\mathbb{Q}$ is not an injective $\mathbb{Z}$-module.

(20%) **6**. Let M be an abelian group. We view M as a $\mathbb{Z}$-module. Prove or disprove that M is isomorphic to a submodule of an injective $\mathbb{Z}$-module.

(10%) **7**. Let L be an extension field of a field K. Suppose that $y, z \in L$ are transcendental elements over K such that $y + z$ is algebraic over K. Prove or disprove that the simple extension $K(yz)$ of K is a transcendental extension.

(20%) **8**. Let K be an infinite field. Let $K(x)$ be the field of rational functions in x over K. Prove or disprove that $K(x)$ is a Galois extension over K.