2006年9月

歸檔

國立中央大學數學系博士班資格考試試題

科目：代數

1. Describe the Galois groups of the following polynomials:

   (a) $x^3 - 3x + 1$.

   (b) $x^5 - 2$.

   (10 pts)

2. Let $\mathbb{F}_{p^n}$ be a finite field of p^n elements, p is a prime number.

   (a) Explain why $\mathbb{F}_{p^n}/\mathbb{F}_p$ is a finite Galois extension.

   (b) Let $\sigma_p : \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ be defined by $\sigma_p(x) = x^p$. Show that

   $$\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_{p^n}) = < \sigma_p >$$

   (the cyclic group generated by σ_p).

   (10 pts)

3. (a) Give the definition and an example of Noetherian ring.

   (b) State the Hilbert Basis Theorem.

   (c) State the fundamental structure theorem for finitely generated modules over a P.I.D..

   (15 pts)

4. (a) Let $F \subset E$ be an algebraic fields extension. Show that any subring D of E containing F is a subfield of E.

   (b) Let $\mathbb{C}$ be the field of complex numbers. Show that

   $$\bar{\mathbb{Q}} = \{\alpha \in \mathbb{C} | \alpha \text{ is algebraic over } \mathbb{Q}\}$$

   is a subfield of $\mathbb{C}$.

   (10 pts)

5. (1) Is 2 an irreducible element in $\mathbb{Z}[\sqrt{-3}]$? Prove your answer.

   (2) Is 2 a prime element in $\mathbb{Z}[\sqrt{-3}]$? Prove your answer.

   (15 pts)

6. Let matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

(1) Compute A^2, A^3, and A^n.

(2) If $f(t) \in \mathbb{R}[t]$ is such that $f(A)$ is equal to zero matrix, then show that $(t^3 - 4t) | f(t)$.

(10 pts)

7. Let G be a finite group of order 45. Then

(1) Find the numbers of Sylow 3-subgroup, Sylow 5-subgroup of G.

(2) Show that G is a direct product of two normal subgroups of G.

(3) Show that G is an abelian group.

(15 pts)

8. Let G be a group of order 10. Then show that

(1) There exists two elements $a, b \in G$ such that $o(a) = 2, o(b) = 5$.

(2) (b) is a normal subgroup of G.

(3) If $aba^{-1} = b^n$ where $0 \leq n \leq 4$, then compute $n =$?

(15 pts)