Algebra

February 12, 2001

1. Find counter examples and explain what is wrong in the proofs:

   (a) (5 pts.) Let $N_1 \subseteq N_2$ be two normal subgroups of G. Then we have that

   $$(G/N_2) \times (N_2/N_1) \cong G/N_1.$$

   proof. Since $N_2 \times N_1 \cong N_1 \times N_2$, it implies that $G/N_2 \times N_2/N_1 \cong (G \times N_2)/(N_2 \times N_1) \cong (G \times N_2)/(N_1 \times N_2) \cong G/N_1 \times N_2/N_2 \cong G/N_1$.

   (b) (5 pts.) Suppose that R is a finite ring of characteristic p, where p is a prime integer. Then $|R| \equiv 1 \pmod{p}$. ($|R|$ is the number of elements in R.)

   proof. Consider the group $\mathbb{Z}/p\mathbb{Z}$ acts on the set R by $\bar{n} \cdot r = nr$ for all $r \in R$, $n \in \mathbb{Z}$ ($\bar{n}$ means the residue of n modulo p). Consider the set $R_0 = \{r \in R \,|\, \bar{n} \cdot r = r, \forall\, \bar{n} \in \mathbb{Z}/p\mathbb{Z}\}$. Since p is a prime, it implies that $R_0 = \{0\}$. Therefore $|R| \equiv |R_0| = 1 \pmod{p}$ and our claim follows.

2. Let H be a normal subgroup of a Group G and let $\pi : G \to G/H$ be the canonical epimorphism (*i.e.* $\pi(g) = gH$).

   (a) (8 pts.) Suppose that P is a Sylow p-subgroup of G. Prove that $\pi(P)$ is a Sylow p-subgroup of G/H.

   (b) (8 pts.) Let $\wp$ be a Sylow p-subgroup of G/H. Show that there exists a Sylow p-subgroup P of G such that $\pi(P) = \wp$.

   (c) (9 pts.) Suppose that for any Sylow p-subgroup P of G, $hPh^{-1} = P$, $\forall\, h \in H$. Prove that π gives a one-to-one correspondence between the set of all Sylow p-subgroups of G and the set of all Sylow p-subgroups of G/H.

3. (a) (5 pts.) Let $\mathbb{R}$ be the field of real numbers. Prove that any polynomial $f(x) \in \mathbb{R}[x]$ can be factorized as a product of polynomials in $\mathbb{R}[x]$ of degree 1 or 2.

   (b) (10 pts.) Let K be a finite extension of degree n over a field k and suppose that $\mathrm{char}(k) \nmid n$, if $\mathrm{char}(k) \neq 0$. Prove that K is an algebraic closure of k if and only if every polynomial $f(x) \in k[x]$ can be factorized as a product of polynomials in $k[x]$ of degree dividing n.

4. Let K be a field and let $\overline{K}$ be a fixed algebraic closure of K. In the following questions, F_1, F_2, K_1 and K_2 are fields in $\overline{K}$. Suppose that F_1 is a Galois extension of K_1 and F_2 is a Galois extension of K_2. Prove or give a counter example of the following:

   (a) (7 pts.) $F_1 \cap F_2$ is a Galois extension of $K_1 \cap K_2$.

   (b) (8 pts.) F_1F_2 is a Galois extension of K_1K_2.

5. Let R be the polynomial ring $\mathbb{Z}[x]$. Fix a prime number p in $\mathbb{Z}$. Suppose that $f(x) = x^n + \sum_{i=0}^{n-1} a_i x^i \in \mathbb{Z}[x]$, where $p \mid a_i$ for all $0 \leq i \leq n-1$. Denote the ring $\mathbb{Z}[f(x)]$ by D.

   (a) (5 pts.) Prove that R is integral over D and prove that D is integrally closed.

   (b) (5 pts.) Prove that R is a free D-module with basis $\{1, x, \ldots, x^{n-1}\}$.

   (c) (5 pts.) Prove that the ideal $(p, f(x))$ generated by p and $f(x)$ in D is a maximal ideal.

   (d) (5 pts.) Consider the polynomial $F(T) = f(T) - f(x)$ in $D[T]$. Prove that $F(T)$ is irreducible in $D[T]$.

   (e) (7 pts.) Let K be the quotient field of D and L be the splitting field of $F(T)$ over K. Prove that $x \in L$ and $[L : K] > n$.

   (f) (8 pts.) Let Γ be the Galois group of L over K. Suppose that $g(x) \in R$ satisfies $g(x^\tau) = g(x)$, $\forall \tau \in \Gamma$. Prove that there exists a polynomial $h(x) \in \mathbb{Z}[x]$ such that $g(x) = h(f(x)) \in D$.