歸檔

中央大學數學系博士生入學考：分析

May 15, 2009

第一題每小題 10 分，其餘各題 12 分，共 100 分。

1. Prove or disprove the following statements:
   (1) The image of a mesurable set under a continuous transformation is measurable.
   (2) Let f be a Lebesgue integrable funtion on $[a,b] \subseteq \mathbb{R}$. Then f is Riemann integral.
   (3) Let f and $\{f_n\}$ be measurable functions which are defined and finite a.e. in a set E. If $f_n \to f$ a.e. on E, then $\{f_n\}$ converges in measure on E to f.
   (4) If f is absolutely continuous on $[a,b] \subseteq \mathbb{R}$, then it is of bounded variation on $[a,b]$.

2. Let $f \in L(0,1)$. Show that $x^k f(x) \in L(0,1)$ for $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} \int_0^1 x^k f(x)\, dx = 0.$$

3. Let $\phi(x), x \in \mathbb{R}^n$, be a bounded measurable function such that $\phi(x) = 0$ for $|x| \geq 1$ and $\int \phi = 1$. For $\varepsilon > 0$, let $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(x/\varepsilon)$. If $f \in L(\mathbb{R}^n)$, show that

$$\lim_{\varepsilon\to 0}(f * \phi_\varepsilon)(x) = f(x) \qquad \text{in the Lebesgue set of } f.$$

4. Show that if $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, then

$$\lim_{|h|\to 0} \|f(x+h) - f(x)\|_p = 0.$$

5.Let $f \in L(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Prove that $f \in L^p(\mathbb{R}^n), 1 < p < 2$ and $\|f\|_p \leq C(\|f\|_1 + \|f\|_2)$, where C depends only on p and n.

6. Let $\{\phi_k\}$ be any orthonormal system for $L^2(\mathbb{R}^n)$ and $f \in L^2(\mathbb{R}^n)$. Set $c_k = \langle f, \phi_k\rangle$. Show that $\{c_k\} \in \ell^2$ and $\|\{c_k\}\|_{\ell^2} \leq \|f\|_2$.