Mathematical Statistics

1. Let X_{i1}, X_{i2} be independent observations from $n(\mu_i, \sigma^2)$, $i = 1, 2, \cdots, n$, where μ_i and σ^2 are all unknown.

   (a) Find the maximum likelihood estimator (MLE) of σ^2. (5%)

   (b) Is the MLE in (a) consistent for σ^2? Justify your answer. (8%)

2. Let $X_1, X_2, \cdots, X_n$ be a random sample from a uniform distribution over $(\theta-1, \theta+1)$.

   (a) Find minimal sufficient statistics of θ. (5%)

   (b) Is the statistics in (a) complete? Justify your answer. (8%)

3. Assume that U is an unbiased estimator of θ and let $T = t(\underset{\sim}{X})$ be sufficient for θ.

   (a) Show that $V = E(U|T)$ is a statistic. (5%)

   (b) Prove that V is unbiased for θ and the variance of V is not larger than that of U. (8%)

4. Let $X_1, X_2, \cdots, X_n$ be a random sample from a normal distribution $n(\mu, 1)$, where $-\infty < \mu < \infty$.

   (a) Show that $T = \exp(\overline{X})$ is the uniformly minimum variance unbiased estimator (UMVUE) of $\exp(\mu + 1/2n)$, where $\overline{X} = (X_1 + \cdots + X_n)/n$. (5%)

   (b) Find the variance of T. Does it reach its Cramer-Rao lower bound? Justify your answer. (8%)

5. Let $X_1, X_2, \cdots, X_n$ be a random sample from the distribution with pdf

$$\begin{aligned} f(x;\theta) &= \theta x^{\theta-1} \quad , \quad 0 < x < 1 \\ &= 0 \qquad\quad , \quad \text{otherwise}, \end{aligned}$$

with $\theta > 0$.

(a) Let $F(x;\theta)$ be the cdf of X. Show that $T(X_1, \cdots, X_n, \theta) = -\sum_{i=1}^{n} \ell n F(X_i;\theta)$ is a pivot quantity. *(8%)*

(b) Constract a $100(1-\alpha)\%$ confidence interval for θ based on $T(X_1, \cdots, X_n, \theta)$ in (a). Be specific about how you obtain the endpoint of the interval. *(8%)*

6. Let $X_1, X_2, \cdots, X_n$ be a random sample from the distribution with pdf

$$\begin{aligned} f(x;\theta) &= e^{-(x-\theta)} \quad , \quad \theta \le x < \infty \\ &= 0 \qquad\quad\;\; , \quad \text{otherwise}, \end{aligned}$$

with $\theta \in (-\infty, \infty)$.

(a) Find the most simplified form of the level α likelihood ratio test of H_0: $\theta = 0$ versus H_1: $\theta \neq 0$. *(8%)*

(b) Find the power function for the test in (a). *(8%)*

(c) Is the test in (a) unbiased? Justify your answer. *(8%)*

7. Let $X_1, X_2, \cdots, X_n$ be a random sample from $n(\mu, \sigma^2)$, where σ^2 is unknown. Find a minimum value for n to guarantee with probability 0.90, that a 95% confidence interval for μ will have length no more than $\sigma/4$. *(8%)*