2008 PhD Program Entrance Examination

Numerical Analysis

1. (15%) Derive directly the formula of Newton's method for solving the nonlinear system

$$\begin{cases} f(x,y) &= 0, \\ g(x,y) &= 0, \end{cases}$$

and starting with $(0,1)$, perform two iterations of Newton's method on the following system:

$$\begin{cases} 4x^2 - y^2 &= 0, \\ 4xy^2 - x &= 1. \end{cases}$$

2. (15%) Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a given symmetric and positive definite matrix, and $\mathbf{b} \in \mathbb{R}^n$ be a given vector. Let $0 < \lambda_{\min} \le \lambda_2 \le \cdots \le \lambda_{n-1} \le \lambda_{\max}$ be the n real eigenvalues of $\mathbf{A}$.

   (a) Show that the condition number $\mathcal{K}(\mathbf{A})$ of the nonsingular matrix $\mathbf{A}$ relative to the norm $\|\cdot\|_2$ is

$$\mathcal{K}(\mathbf{A}) = \frac{\lambda_{\max}}{\lambda_{\min}}.$$

   (b) Suppose that $\widetilde{\mathbf{x}}$ is an approximation to the solution of $\mathbf{Ax} = \mathbf{b}$. Show that

$$\frac{\|\mathbf{x} - \widetilde{\mathbf{x}}\|_2}{\|\mathbf{x}\|_2} \le \left(\frac{\lambda_{\max}}{\lambda_{\min}}\right) \frac{\|\mathbf{r}\|_2}{\|\mathbf{b}\|_2}, \quad \text{provide } \mathbf{x} \neq \mathbf{0} \text{ and } \mathbf{b} \neq \mathbf{0},$$

   where $\mathbf{r}$ is the residual vector for $\widetilde{\mathbf{x}}$.

3. (15%) Let $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{n\times n}$, $a_{ii} \neq 0$, for $i = 1, \cdots, n$, and $\mathbf{b} = (b_1, \cdots, b_n)^t \in \mathbb{R}^n$. The basic concept of iterative methods for solving the system of linear equations $\mathbf{Ax} = \mathbf{b}$ is to convert the system into an equivalent system of the form $\mathbf{x} = T\mathbf{x} + \mathbf{c}$ and lead to the iterations

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}.$$

   (a) Find T and $\mathbf{c}$ for the Gauss-Seidel method.

   (b) Please give a necessary and sufficient condition to ensure the convergence of the Gauss-Seidel method for any initial guess $\mathbf{x}^{(0)} \in \mathbb{R}^n$.

4. (15%) Let f be sufficiently smooth and satisfy the Lipschitz condition such that there exists a unique solution $y(t)$, $a \le t \le b$, of the following initial value problem

$$y'(t) = f(t,y) \text{ for } a \le t \le b, \quad y(a) = y_a \in \mathbb{R}.$$

   A second-order Runge-Kutta method for the numerical approximation of the problem can be written as

$$\begin{aligned} w_{i+1} &= w_i + h\{a_1 f(t_i, w_i) + a_2 f(t_i + \alpha, w_i + \beta f(t_i, w_i))\}, \\ w_0 &= y_a. \end{aligned}$$

   Given $a_1 = 1/2$, determine the values of a_2, α and β, and determine the rate of convergence of the method.

5. (20%)

(a) Assume that $u \in C^4[x_0 - h, x_0 + h]$. Use Taylor's theorem and intermediate value theorem to derive the following formulas for approximating $u'(x_0)$ and $u''(x_0)$:

$$\begin{aligned} u'(x_0) &= \frac{1}{h}\Big\{u(x_0) - u(x_0 - h)\Big\} + \frac{h}{2}u''(\xi), \quad \text{for some } \xi \in (x_0 - h, x_0); \\ u''(x_0) &= \frac{1}{h^2}\Big\{u(x_0 - h) - 2u(x_0) + u(x_0 + h)\Big\} - \frac{h^2}{12}u^{(4)}(\eta), \\ &\qquad\qquad \text{for some } \eta \in (x_0 - h, x_0 + h). \end{aligned}$$

(b) Consider the following two-point boundary value problem:

$$\begin{cases} -\varepsilon u''(x) + u'(x) = 0 \ \text{ for } \ 0 < x < 1, \\ u(0) = 1 \ \text{ and } \ u(1) = 0, \end{cases}$$

where ε is a given positive constant. Let $0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1$ be a uniform partition of $[0, 1]$ with mesh size $h > 0$. Applying the numerical differentiation formulas derived in part (a) to approximate the solution of the boundary value problem, we can obtain a linear system of the form $\mathbf{AU} = \mathbf{b}$, where $\mathbf{U} = (U_1, U_2, \cdots, U_{N-1})^\top$, $U_0 = 1$, $U_N = 0$, and U_i denotes the approximation to $u(x_i)$. Find the $(N-1) \times (N-1)$ matrix $\mathbf{A}$ and $(N-1) \times 1$ vector $\mathbf{b}$.

6. (20%) Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be an open bounded domain with smooth boundary $\partial\Omega$. Consider the following boundary value problem:

$$\begin{aligned} -\Delta u + u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $f \in L^2(\Omega)$.

(a) Give a variational formulation for the problem using the Hilbert space $V = H_0^1(\Omega)$ and prove the variational problem has a unique solution in V.

(b) Let $V_h \subseteq V$ be a finite-dimensional finite element space. Give the finite element formulation for the problem and show that the formulation is equivalent to a linear system $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is a symmetric and positive definite matrix.