中央大學數學系博士班入學考【圖論】 2008/5/16

Problem 1. Let G be a graph. The chromatic polynomial $\chi(G;k)$ of G is defined to be a function of k. An acyclic orientation of G is an orientation having no directed cycle. Show that the value of $\chi(G;k)$ at $k=-1$ is $(-1)^{n(G)}$ times the number of acyclic orientations of G, where $n(G)$ is the number of vertices of G. (Hint: you can use induction on the number of edges of G) (10pt)

Problem 2. Does there exist a graph G with the chromatic polynomial $\chi(G;k)=k^4-3k^3+3k^2$? (10pt)

Problem 3. (a) Prove Euler's Formula: if a connected plane graph G has exactly n vertices, e edges, and f faces, then n-e+f = 2. (10pt)
(b) Show that there are exactly five Platonic solids. (5pt)

Problem 4. Let G be a k-colorable graph, and let P be a set of vertices in G such that $d(x,y)\geq 4$ whenever $x,y\in P$. Prove that every coloring of P with colors from $\{1,2,3,\cdots,k+1\}$ extends to a proper (k+1)-coloring of G. (10pt)

Problem 5. Show that if G is a bipartite graph and λ is an eigenvalue of the adjacency A of G with multiplicity m, then $-\lambda$ is also an eigenvalue of A with multiplicity m. (10pt)

Problem 6. Show that planar graphs are 5-colorable. (10pt)

Problem 7. Prove that a graph is bipartite if and only if it has no odd cycle. (10pt)

Problem 8. Let T be an m-vertex tree. Let Ramsey number $R(T,K_n)$ be the smallest integer t such that every (red, blue)-edge-coloring of K_t contains a red T or a blue K_n.
(a) Show that $R(T,K_n)=(m-1)(n-1)+1$. (10pt)
(b) Show that $R(K_3,K_3)=6$. (10pt)
(c) Show that $R(K_3,K_4)=9$. (5pt)