中央大學數學系博士生入學考：分析

May 16, 2008

1. Let $E \subseteq \mathbb{R}$. Prove or disprove

(1) If E satisfies, for any open subset G of $\mathbb{R}$

$$|G \cap E|_e + |G \backslash E|_e = |G|_e \qquad (|\cdot|_e \text{ is Lebesgue outer measure}),$$

then E is a Lebesgue measurable set. (10%)

(2) If $|E|_e = 0$, then the set $\{x^3 : x \in E\}$ is a null Lebesgue measurable set.(10%)

2. Let f be real-valued function on $[a, b], -\infty < a < b < \infty$. Prove or disprove:

(a) If f is absolutely continuous function, then f satisfies a Lipschitz condition. (8%)

(b) If f is absolutely continuous function, then f is of bounded variation.(8%)

3. Let f be a function with measurable domain $E \subseteq \mathbb{R}^n$. Show that f is measurable iff the function g defined by $g(x) = f(x)$ for $x \in E$ and $g(x) = 0$ for $x \notin E$ is measurable.(16%)

4. Let $1 \leq p < \infty$ and $f, \{f_k\} \in L^p$. Show that if $f_k \to f$ almost everywhere and $\|f_k\|_p \to \|f\|_p$, then $\|f_k - f\|_p \to 0$.(16%)

5. Suppose the linear operator T satisfies $\|Tf\|_\infty \leq C_1 \|f\|_\infty$ and

$$|\{x \in \mathbb{R}^n : |Tf(x)| > \alpha\}| \leq \frac{C_2}{\alpha} \|f\|_1 \qquad (\alpha > 0),$$

where C_1 and C_2 are independent of f and $|\cdot|$ denotes by Lebesgue measure.

Then, for $1 < p < \infty$, we have

$$\|Tf\|_p \leq C \|f\|_p,$$

where C depends only on C_1, C_2 and p.(16%)

6. Let T be a convolution operator $Tf(x) = \int_{\mathbb{R}^n} f(y) k(x - y) dy$ with $k > 0$. If

$$\|Tf\|_p \leq C \|f\|_p, \ 1 < p < \infty, \qquad \text{for} \quad f \in L^p(\mathbb{R}^n),$$

show that $\|Tf\|_{p'} \leq C \|f\|_{p'}, 1/p + 1/p' = 1$. (16%)