ALGEBRA

May 2008

1. (25 %)

(a) Show that there exists a simple group of order 60.

(b) Classify groups of order 45.

2. (25 %)

(a) Let R be an integral domain and suppose that every prime ideal in R is principal. Show that R is a PID. (Hint: consider the set of ideals of R that are not principal and use Zorn's lemma.)

(b) Prove that a UFD is a PID if and only if every nonzero prime ideal is a maximal ideal.

3. (20 %)

(a) Let R be an integral domain but not a field and F be the quotient field of R. Show that the set of R-module homomorphisms from F to R is trivial, that is,

$$\mathrm{Hom}_R(F, R) = \{0\}.$$

(b) Let R be a commutative ring with 1 and M be any R-module. Prove that $\mathrm{Hom}_R(\cdot, M)$ is a left exact functor between the categories of R-modules, but it is in general not a right exact functor.

4. (30 %)

(a) Let $f(x) \in \mathbf{Q}[x]$ be irreducible of degree p with p a prime. If $f(x)$ has exactly $p-2$ real roots and 2 complex roots, prove that the Galois group of $f(x)$ over $\mathbf{Q}$ is the symmetric p group S_p. (Here, $\mathbf{Q}$ is the field of rational numbers.)

(b) Let k be a field and $f(x) \in k[x]$ be a separable polynomial. Assume that $f(x) = g(x)h(x)$ in $k[x]$. Find an example of $f(x)$ such that the Galois groups of $f(x)$, $g(x)$ and $h(x)$ over k are nontrivial and the Galois group of $f(x)$ over k is isomorphic to the direct product of the Galois groups of $g(x)$ and $h(x)$ over k. (Justify your answer) Moreover, prove that the Galois group of $(x^3-2)(x^3-3)$ over $\mathbf{Q}$ is not isomorphic to the direct product of the Galois groups of x^3-2 and x^3-3 over $\mathbf{Q}$.