國立中央大學數學系 96 學年度博士班入學考試筆試試題

歸檔

科目：分析

NCU PHD PROGRAM ENTRANCE EXAM: ANALYSIS

(May 18, 2007)

Note: All functions are real-valued.

(1) Let $(X, \mathcal{B}, \mu)$ be a measure space and $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued measurable functions on X. Show that $\{x \in X : \lim_{n\to\infty} f_n(x) \text{ exists}\}$ is a measurable set. (10%)

(2) Let

$$f(x) = \begin{cases} x\sin(\frac{1}{x}) & \text{if } 0 < x \le 1, \\ 0 & \text{if } x = 0; \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x^3\sin(\frac{1}{x}) & \text{if } 0 < x \le 1, \\ 0 & \text{if } x = 0. \end{cases}$$

(a) Determine whether f and g are of bounded variation on $[0, 1]$. Give your proof in each case. (9%)

(b) Determine whether f and g are absolutely continuous on $[0, 1]$. Give your proof in each case. (6%)

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be integrable with respect to the Lebesgue measure. Show that the fuunction $g : [0, \infty) \to \mathbb{R}$ defined by

$$g(t) = \sup\{\int_{\mathbb{R}} |f(x + y) - f(x)|dx : |y| \le t\}$$

for $t \ge 0$ is continuous at $t = 0$. (15%)

(4) If a Lebesgue integrable function $f : [0, 1] \to \mathbb{R}$ satisfies $\int_{[0,1]} x^{2n} f(x)dx = 0$ for each $n = 0, 1, 2, \ldots$, then show that $f = 0$ almost everywhere. (15%)

(5) Let $1 \le p < \infty$ and $f \in L^p[0, 1]$. Recall that f will be considered defined on all of $\mathbb{R}$ by $f(t) = 0$ if $t \notin [0, 1]$. For each $h > 0$, we define

$$f_h(t) = \frac{1}{2h} \int_{[t-h,\ t+h]} f(x)dx$$

for each $t \in [0, 1]$.

(a) Prove that f_h is continuous on $[0, 1]$. (10%)

(b) Prove that $\|f_h\|_p \le \|f\|_p$. (10%)

(6) Let $(C[0, 1], \|\cdot\|_1)$ be the normed space of all continuous real-valued functions on $[0, 1]$ with the norm $\|f\|_1 = \int_{[0,1]} |f|$, and

$$A = \{f \in C[0, 1] : f(0) = 0, f \text{ is differentiable on } (0, 1) \text{ and } |f'| \le 1\}.$$

Prove that the integral equation

$$f(x) = \frac{1}{2} \int_0^x e^{-xy} f(y)dy, \quad x \in [0, 1],$$

has a unique solution $f \in \overline{A}$, where $\overline{A}$ is the closure of A in $(C[0, 1], \|\cdot\|_1)$. (15%)

(7) Let $1 < p < q < \infty$ and S be a closed vector subspace of the normed space $(L^p[0, 1], \|\cdot\|_p)$. Suppose that S is also contained in $L^q[0, 1]$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in S. Prove that $\{f_n\}_{n=1}^{\infty}$ is convergent in $(L^q[0, 1], \|\cdot\|_q)$ if and only if $\{f_n\}_{n=1}^{\infty}$ is convergent in $(L^p[0, 1], \|\cdot\|_p)$. (10%)