中央大學博士資格考--實變-NCU Ph.D Qualify Exam, Real Analysis.

August 2024

Grades are assigned not only based on the correctness of the solutions but also on your presentation. There are **seven** problems on this exam.

1. **(10ps) Prove or Disprove:** Suppose $\forall\, A, B \subseteq \mathbb{R}^n$ with $A \cap B = \emptyset$ we have $|A \cup B|_e = |A|_e + |B|_e$ then all sets in $\mathbb{R}^n$ are measurable. (The notation $|A|_e$ denotes the Lebesgue outer measure of the set A.)

2. **(15ps) Prove or Disprove:** $\exists\ f \in L^2([0,\pi])$ such that both the inequalities
$$\int_0^\pi (f(x) - cos(x))^2\, dx \leq \frac{1}{9} \quad \text{and} \quad \int_0^\pi (f(x) - sin(x))^2\, dx \leq \frac{4}{9}$$
hold? Justify your answer.

3. **(15ps)** Determine (with a proof) the value of $|[0,1] \setminus \mathcal{N}|_e$ where $\mathcal{N} \subseteq [0,1]$ is any non-measurable set.

4. **(15ps)** Suppose $f \in L(E)$ (that is, f is Lebesgue integrable over E) prove that
$$\forall \epsilon > 0\ :\ \exists \delta > 0\ :\ \forall A \subseteq E,\ |A| < \delta \implies \int_A |f| < \epsilon\ .$$

5. **(15ps)** Suppose $f, f_k : E \to [0,\infty]$ are measurable and $f_k \xrightarrow{m} f$ (convergence in measure). Prove that
$$\int_E f \leq \liminf_{k\to\infty} \int_E f_k\ .$$

6. **(15ps)** Find the limit $\lim_{n\to\infty} n\ sin(2\pi n!\, e)$.

7. **(15ps)** Let $A \subseteq \mathbb{R}$ be such that $|A|_e > 0$. Prove that
$$\forall\ 0 < \delta < 1\ :\ \exists \text{ an interval } I \text{ such that } \quad |A \cap I|_e \geq \delta |I|\ .$$