Qualifying Exam for Differential Equations

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2024 Winter

Problem 1. (20pts) Let $D \subseteq \mathbb{R} \times \mathbb{R}^n$ be an open set, $(t_0, x_0) \in D$, and $f : D \to \mathbb{R}^n$ be a function. For a given interval $I \subseteq \mathbb{R}$ containing t_0 as an interior point, a function ϕ is said to be an ε-approximate solution of

$$\frac{dx}{dx} = f(t, x(t)), \qquad x(t_0) = x_0, \tag{0.1}$$

on the interval I if

1. $\phi : I \to \mathbb{R}^n$ is piecewise C^1;
2. $(t, \phi(t)) \in D$ for all $t \in I$;
3. $\left|\phi'(t) - f(t, \phi(t))\right| < \varepsilon$ for all $t \in I$.

Prove that for any sufficiently small $\varepsilon > 0$, there exists an ε-approximate solution of (0.1) on the interval $I = \left\{t \,\middle|\, |t - t_0| \leqslant C\right\}$, where $C = \min\left\{a, \dfrac{b}{M}\right\}$.

Problem 2. (15pts) Let $x_1(t)$, $x_2(t)$ be differentiable functions defined on $[a, b]$ such that for all $t \in [a, b]$,

1. $|x_1(a) - x_2(a)| \leqslant \delta$,
2. $|x_1'(t) - f(t, x_1(t))| \leqslant \varepsilon_1$, and
3. $|x_2'(t) - f(t, x_2(t))| \leqslant \varepsilon_2$.

Show that if there is a constant $L > 0$ such that

$$|f(t, x_1) - f(t, x_2)| \leqslant L|x_1 - x_2| \qquad \forall\, x_1, x_2 \in \mathbb{R}\,,$$

then for all $t \in [a, b]$,

$$|x_1(t) - x_2(t)| \leqslant \delta e^{L(t-a)} + (\varepsilon_1 + \varepsilon_2)\frac{e^{L(t-a)} - 1}{L}\,.$$

Problem 3. (15pts) Analyze the stability properties of the trivial solution of the following system:

$$\begin{aligned} x' &= -a_0 f(x) - \sum_{i=1}^{n} a_i z_i\,, \\ z_i' &= -\lambda_i z_i + b_i f(x) \quad \text{for } 1 \leqslant i \leqslant n\,, \end{aligned}$$

where a_i, λ_i, and b_i are all positive and $xf(x) > 0$ if $x \neq 0$.

Hint: Choose $v(x, z) = \displaystyle\int_0^x f(s)\, ds + \frac{1}{2}\sum_{i=1}^{n}\left(\frac{a_i}{b_i}\right) z_i^2$.

Problem 4. (10pts) Consider the Predator-Prey model

$$x_1' = a_1 x_1 - b_1 x_1 x_2 \,,$$
$$x_2' = -a_2 x_2 + b_2 x_1 x_2 \,,$$

where a_1, a_2, b_1, and b_2 are all positive. Find all equilibrium and determine their stability properties.

Problem 5. (15pts) Analyze the stability properties of the trivial solution of the following equation:

$$x' = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 4 & 0 \\ -1 & 0 & 4 \end{bmatrix} x + \begin{bmatrix} x_1 x_2 \\ x_1 x_3 \\ \sin(x_1 x_2 x_3) \end{bmatrix},$$

where $x = [x_1\, x_2\, x_3]^{\mathrm{T}}$.

Problem 6. (10pts) Show that the equation

$$x'' + (3x^5 - 8x^3 - 12x)x' + x = 0$$

has a unique nontrivial periodic solution.

Problem 7. Let $\phi(t;\varepsilon)$ be the limit cycle solution of

$$y'' + \varepsilon(y^2 - 1)y' + y = 0\,, \qquad y(0) = y_0 > 0, y'(0) = 0\,.$$

1. (8pts) Show that $\phi(t;\varepsilon) \to \alpha \cos t$ as $\varepsilon \to 0$ for some constant $\alpha > 0$.
2. (7pts) Compute the value of α.