1. Let $X_1, \cdots, X_n$ denote a sequence of independent and identically distributed Bernoulli random variables with

$$P(X_i = 1) = p, P(X_i = 0) = 1 - p, i = 1, \cdots, n, 0 < p < 1,$$

where p is unknown.

(a) (5%) Find the maximum likelihood estimator of p.

(b) (10%) Find the uniformly minimum variance unbiased estimator of p.

(c) (10%) Find a Bayes estimator of p.

(d) (10%) Show that the three estimators obtained above all converge in probability to p as n goes to infinite.

(e) (10%) Calculate $Cov(\bar{X}, S^2)$, where

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \text{ and } S^2 = \frac{1}{n}\sum_{i=1}^{n}[X_i - \bar{X}]^2.$$

2. Let $X_1, \cdots, X_n$ denote a sequence of independent and identically distributed normal random variables with

$$EX_i = \mu_x, VarX_i = \sigma^2, i = 1, \cdots, n, -\infty < \mu_x < \infty, 0 < \sigma^2 < \infty.$$

(a) (10%) Assume that $\sigma^2 = 1$. Find the uniformly most powerful test for

$$H_0 : \mu_x = 0 \text{ vs } H_1 : \mu_x > 0.$$

Moreover, let $Y_1, \cdots, Y_n$ denote a sequence of independent and identically distributed normal random variables with

$$EY_i = \mu_y, VarY_i = \sigma^2, i = 1, \cdots, n, -\infty < \mu_y < \infty, 0 < \sigma^2 < \infty.$$

(b) (15%) Find the generalized likelihood ratio test for

$$H_0 : \mu_x = \mu_y, \text{ unspecified, and } \sigma^2 \text{ unspecified,}$$

against all alternatives.

3. Let $X_1, \cdots, X_n$ denote a sequence of independent and identically distributed Cauchy random variables with probability density function

$$f(x, \theta) = \frac{1}{\pi[1 + (x - \theta)^2]}, -\infty < x, \theta < \infty.$$

(a) (10%) Find a level α confidence interval of θ.

(b) (10%) Find $Ee^{itX_1}, i = \sqrt{-1}, -\infty < t < \infty$.

4. (10%) Consider the sum $HS(n, L)$ of squares of horizontal distances between each point of $\{(X_1, Y_1), \cdots, (X_n, Y_n)\}$ and any line L. Let $y = \gamma x + \delta$ denote the line L_h such that $HS(n, L) \geq HS(n, L_h)$, find γ and δ.