PHD Entrance Examination–Algebra–05/18/2012

(20%) 1. Let p be a prime number and let G be a finite group with $p \mid o(G)$. Suppose that H is subgroup of order p.
(a) Prove or disprove that H is contained in some p-Sylow subgroup of G.
(b) Suppose further that H is normal in G. Prove or disprove that H is contained in every p-Sylow subgroup of G.

(10%) 2. Prove or disprove that every subgroup of a solvable group is solvable.

(20%) 3. Let A be a commutative ring. Suppose that $1 \neq 0$ in A. Let S be a multiplicative set of A not containing 0. Let $\mathfrak{p}$ be a maximal element in th set of ideals of A whose intersection with S is empty.
(a) Prove or disprove that $\mathfrak{p}$ is prime.
(b) Prove or disprove that $\mathfrak{p}$ is maximal.

(20%) 4. Consider the polynomial ring $\mathbb{Z}[x, y]$ in two variables and the principal ideal I generated by $xy - 1$. Denote R to be the quotient ring $\mathbb{Z}[x, y]/I$.
(a) Prove or disprove that $(R, +)$ is a free $\mathbb{Z}$-module.
(b) Prove or disprove that $(R, +)$ is a free $\mathbb{Z}[x]$-module.

(10%) 5. Let K be the splitting field of the polynomial $f(x) = x^4 - 2$ over $\mathbb{F}_5$, where $\mathbb{F}_5$ is the finite field with 5 elements. Determined all the subfields of K. Prove your answer.

(20%) 6. Let E be a finite extension over k and let F be any extension over k.
(a) Suppose that E is Galois over k. Prove or disprove that the compositum EF is Galois over F.
(b) Suppose that the compositum EF is Galois over F. Prove or disprove that E is Galois over k.

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